TOPICS IN COMPUTABLE MODEL THEORY

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This dissertation contains several results in computable model theory and in the theory of automatic structures. The first chapter is an introduction. In chapter 2 we prove the existence of an uncountably categorical theory whose only computably presentable model is the saturated one. In chapter 3 we construct a computably categorical saturated model, and in chapter 4 we show that there exists a prime model of finite computable dimension. In chapter 5 we study non-computable presentations of Π^0_1-algebras. The last chapter is devoted to automatic structures. We provide new examples of automatic indecomposable torsion-free abelian groups and construct a new automatic presentation of the group $\mathbb{Z} \times \mathbb{Z}$.

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Chapter 1
General Introduction

This thesis contains several results in the fields of computable model theory and automatic structures which are the parts of a broader area of research called computable mathematics. Computable mathematics studies the effective content of theorems, notions, and constructions from various branches of mathematics, especially from algebra, analysis, combinatorics, and model theory. One of the general goals of computable mathematics is to find out whether the effective versions of classical mathematical results are true (in various senses of what it means to be effective). In the case when the effective version of a theorem is false, we would like to know the limit at which it is still true and to find a sharp counterexample.

We will mainly be concerned with computable model theory. The main objects of study here are computable structures (models) or structures computable relative to some oracle.

Definition 1.1. A structure is called *computable* if its domain is a computable subset of natural numbers and all its functions and relations are uniformly computable. A structure is called *computably presentable* if it is isomorphic to a computable structure.

The first attempt to study computable algebraic structures probably dates back to van der Waerden [72, 73] in 1930, who introduced and studied the notion of an *explicitly given* field. These are the ones whose elements are presented by distinguishable symbols in such a way that one can perform basic field operations effectively. However, this definition was not precise since it did not specify what it meant "to perform operations effectively". Fröhlich and Shepherdson [14, 15] used the rigorous notion of a recursive function to make this definition precise and studied explicit fields in a formal setting. Rabin [69, 70] initiated the study of computable groups and continued the research on computable fields. In the early 1960's Mal'cev [58] introduced the notion of a *constructive* structure, which is equivalent to the concept of a computable structure. Ershov [10] and Goncharov [19, 20, 21, 22] continued the systematic study of constructive models ([11] contains a good survey on this topic). Nerode and Metakides [60] first began to use priority methods from recursion theory in the study of computable structures.

In computable model theory one usually investigates the structure of computable models of a first order theory. Typical questions include the following. Given a first order theory, does it have a computable model? Which models of the theory are computably presentable and which are not? Are there computable models with some specific model-theoretic or computability-theoretic properties?

Many well-known theorems in classical model theory have effective versions. Here are some typical examples. The effective version of Completeness Theorem states

Theorem 1.2. *Every decidable theory has a decidable model.*

A *theory* here is a consistent set of first order sentences closed under logical consequences. A structure $\mathcal{A}$ is *decidable* if its full diagram is computable, that is, there is an algorithm to decide whether $\mathcal{A} \models \varphi(\bar{c})$ for every formula $\varphi(\bar{x})$ and tuple $\bar{c} \in |\mathcal{A}|$. The proof of this theorem uses the fact that the usual Henkin construction can be carried out effectively if the given theory is decidable. The same argument shows that every consistent theory T has a model decidable in T. Goncharov/Nurtazin [24] and independently Harrington [27] proved the following result about the existence of decidable prime models.

Theorem 1.3. *Let T be a complete decidable theory that has a prime model. Then T has a decidable prime model if and only if the set of all principal types of T is computable.*

Morley [63] and Millar [61] gave a characterization of theories that have decidable saturated models.

Theorem 1.4. *Let T be a complete theory. Then T has a decidable saturated model if and only if there is a computable list of all types of T.*

However, many other theorems of classical mathematics do not hold effectively. For instance, the effective version of König's Lemma would state that every infinite computable finitely branching tree has an infinite computable path. This statement is not true since there is an example of infinite computable binary tree with no infinite computable path.

A good introduction to computable mathematics, especially to computable algebra, model theory and combinatorics, as well as an overview of modern developments in this fields can be found in the *Handbooks of recursive mathematics* [12] and in the *Handbook of computability theory* [25]. Journal papers such as Khoussainov/Shore [51], Goncharov/Khoussainov [17] and many other surveys published in *Computability theory and its applications* [8] provide an extensive overview of current trends and open problems in computable mathematics.

The outline of the thesis is as follows. Chapters 2, 3 and 4 are devoted to the study of computable models. In chapter 2 we construct an uncountably categorical theory that has a computable saturated model such that all other models of the theory are not computably presentable. In chapter 3 we provide an example of a computably categorical saturated model whose theory is not $\aleph_0$-categorical. In chapter 4 we show how to construct prime models of finite computable dimension $n > 1$.

In chapter 5 we discuss Σ^0_1- and Π^0_1-structures, notions that generalize the concept of a computable structure. We will study the following question: which computable structures possess non-computable Π^0_1-presentations? We show that many well-known mathematical structures, which fail to have non-computable Σ^0_1-presentations, do possess non-computable Π^0_1-presentations.

In chapter 6 we investigate automatic abelian groups. These are the groups that can be recognized by finite automata. The structures that are recognizable by finite automata are called *automatic*. They form a natural subclass of the computable structures. In fact, all automatic structures are decidable. We will give new examples of automatic torsion-free abelian groups and construct a new automatic presentation of the group $\mathbb{Z}\times\mathbb{Z}$, in which every nontrivial cyclic subgroup is not recognized by a finite automaton.

Below is the description of each chapter in more details.

Chapter 2. A new spectrum of computable models

An important theme in computable model theory is the study of computable models of complete first order theories. More precisely, given a complete first order theory T, one would like to know which models of T have computable copies and which do not. A special case of interest is when T is uncountably categorical. Our goal is to find examples of uncountably categorical theories with new spectra of computable models. This question attracts interest because the known upper bounds on the complexity of the spectra of computable models are quite high (hyperarithmetical in the general case), but the sets which are known to be realized as spectra are very simple.

Recall that a theory T is called *κ-categorical*, where κ is a cardinal, if all models of T of cardinality κ are isomorphic. The well-known result by Morley [62] states that a theory T in a countable language is $\aleph_1$-categorical if and only if it is categorical in every uncountable cardinality. So we will often use the notions *uncountably categorical* and *$\aleph_1$-categorical* interchangeably. Typical examples of $\aleph_1$-categorical theories include the first order theories of the successor structure, an algebraically closed field, and a vector space over a given countable field.

In [3], Baldwin and Lachlan developed the theory of $\aleph_1$-categoricity in terms of strongly minimal sets. They showed that the countable models of an $\aleph_1$-categorical theory T can be listed in an $\omega + 1$ chain

$$\mathcal{A}_0 \preccurlyeq \mathcal{A}_1 \preccurlyeq \cdots \preccurlyeq \mathcal{A}_\omega,$$

where the embeddings are elementary, $\mathcal{A}_0$ is the prime model of T, and $\mathcal{A}_\omega$ is the countable saturated model of T. Based on the theory developed by Baldwin and Lachlan, Harrington [27] and Khisamiev [40] in 1974 proved that if an $\aleph_1$-categorical theory T is decidable then all the countable models of T have computable, and even decidable, presentations. Thus, for decidable $\aleph_1$-categorical theories the question of which models of T have computable presentations is fully settled. However, the situation is far from clear when the theory T is not decidable.

Goncharov in [20] initiated the study of computable models of uncountably categorical theories without the assumption on decidability. He constructed an $\aleph_1$-categorical theory such that its prime model, that is $\mathcal{A}_0$, is computably presentable and the others are not. Kudaibergenov [54] then constructed an $\aleph_1$-categorical theory T such that only $\mathcal{A}_0, \dots, \mathcal{A}_n$ are computably presentable and the others are not. In [47] Khoussainov, Nies, and Shore gave the following definition.

Definition 1.5. Let T be an $\aleph_1$-categorical theory. The *spectrum of computable models* of T, denoted by $\mathrm{SCM}(T)$, is the set

$$\mathrm{SCM}(T) = \{i \in \omega + 1 : \mathcal{A}_i \text{ has a computable presentation}\}.$$

Currently there are two approaches in the study of computable models of $\aleph_1$-categorical theories. One of them addresses the issue on the degree theoretic complexity of models and theories. The other studies the spectrum of computable models. This chapter contributes to the second approach. For completeness of the picture we would like to say a few words about the first approach as well.

The first result here in due to Goncharov and Khoussainov [18]. For every given $n \geqslant 1$, they constructed an $\aleph_1$-categorical theory T of finite signature with the following property. The Turing degree of T is $\mathbf{0}^{(n)}$ and all countable models of T are computable. We mention here that it is still an open problem if there exists an $\aleph_1$-categorical theory of degree $\mathbf{0}^{(\omega)}$ which has a computable model.

To formulate the next results we need to recall a couple of definitions. A complete theory T is *strongly minimal* if every definable (possibly with parameters) subset $B \subseteq A$ of any model $\mathcal{A}$ of T is either finite or cofinite. It is well-known that strongly minimal theories are uncountably categorical [3]. Let $B \subseteq A$ be a subset of a model $\mathcal{A}$. The *algebraic closure* of B, denoted by $\mathrm{acl}(B)$, is the collection of all $a \in A$ that are definable by formulas with parameters from B such that these formulas have finitely many solutions in $\mathcal{A}$. A strongly minimal theory T is *trivial* or has *trivial pregeometry*, if for every subset $B \subseteq A$ of any model $\mathcal{A}$ of T, $\mathrm{acl}(B)$ is equal to the union of $\mathrm{acl}(\{b\})$ for all $b \in B$.

For trivial and strongly minimal theories Goncharov/Harizanov/Laskowski/Lempp/McCoy [23] obtained the following upper bound on the complexity of the models. If T is trivial, strongly minimal and has a computable model, then T is computable in $\mathbf{0}''$. Therefore, all countable models of T are $\mathbf{0}''$-decidable. In particular, they are $\mathbf{0}''$-computable.

Recently, Khoussainov/Laskowski/Lempp/Solomon [53] showed that the upper bound in the above mentioned result is sharp. They constructed a trivial, strongly minimal theory such that its prime model is computable and each of the other countable models computes $\mathbf{0}''$.

As for the spectra of computable models, Nies [64] showed that for every $\aleph_1$-categorical theory T, $\mathrm{SCM}(T)$ is $\Sigma^0_3(\varnothing^\omega)$. It means that in general $\mathrm{SCM}(T)$ may have hyperarithmetical complexity. It turns out that much better upper bounds can be obtained in the cases when T is model complete or when it is trivial and strongly minimal.

A theory T is *model complete* if for any $\mathcal{A}$, $\mathcal{B}$ models of T such that $\mathcal{A}$ is a substructure of $\mathcal{B}$, we have that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$. If T is a model complete $\aleph_1$-categorical theory, then SCM(T) is Σ^0_4 (Nies [64]). Goncharov/Harizanov/Laskowski/Lempp/McCoy [23] showed that if T is trivial and strongly minimal, then SCM(T) is Σ^0_5.

Surprisingly, the known examples of spectra of computable models have very simple descriptions as stated in the next theorem.

Theorem 1.6. *The following sets can be realized as spectra of computable models:*

1) the empty set and $\omega + 1$ *itself* (Harrington [27] and Khisamiev [40])

2) the set $\{0\}$ (Goncharov [20]; Herwig/Lempp/Ziegler [28] showed this for a theory of finite signature)

3) initial segments $\{0, \ldots, n\}$ (Kudaibergenov [54])

4) $(\omega + 1) \setminus \{0\}$ *and* ω (Khoussainov/Nies/Shore [47])

5) intervals $\{1, \ldots, n\}$ (Nies [64]).

Therefore, finding new examples of the spectra of computable models is an interesting problem in computable model theory. In this work we will show that $\{\omega\}$ can be realized as a spectrum.

Theorem 2.3.1. *There exists an* $\aleph_1$*-categorical but not* $\aleph_0$*-categorical theory whose only computably presentable model is the countable saturated one.*

As a tool we introduce a notion related to limitwise monotonic functions, called *S-limitwise monotonic* functions. Using it we construct a family of uniformly computably enumerable sets with specific computability-theoretic properties and then encode it into a computable graph. This graph is a disjoint union of cubes, and each such cube codes a set from the family that we have constructed.

Our notions of A-cubes and S-limitwise monotonic functions naturally generalize the notions of n-cubes and limitwise monotonic functions used in [47] to construct an example of $\aleph_1$-categorical theory T such that all models of T except the prime one have computable presentation.

The material of this chapter is published in Hirschfeldt/Khoussainov/Semukhin [30].

Chapter 3. Computably categorical saturated models

Our main interest in this chapter concerns the existence of a computable non $\aleph_0$-categorical saturated structure with a unique computable isomorphism type. Structures with exactly one computable isomorphism type are called *computably categorical*. In general, the *computable dimension* of the structure is the number of its computable presentations up to computable isomorphism. The known standard examples of computably categorical structures are usually prime models of their own theories or become prime in expansions by finitely many constants. For example, finitely generated computable algebras, the rational numbers under the natural ordering, finite dimensional vector spaces over computable fields and the ring of integers are computably categorical. There are also pathological examples of computably categorical structures that fail to satisfy certain natural properties (for example, existence of Scott families) exhibited by most computably categorical structures [7, 50, 55]. One notes that these specifically constructed computably categorical structures fail to be prime models in expansions by finitely many constants. In fact, the theories of such structures do not have saturated models due to the fact that the theories have uncountably many types.

We are interested in the following question: how does the computable dimension of a structure depend on its model-theoretic properties? In particular, if we consider prime and saturated models, what computable dimensions can these structures have?

We briefly recall some basic notions and facts from model theory. Let T be a complete theory. A countable model of T is said to be *prime* if it is elementarily embeddable into any model of T. A countable model of T is said to be *countable saturated* if the model realizes all the types of the theory in all possible expansions of T by finitely many constants. It is well-known that if T has at most countably many countable models, then T has a countable saturated model as well as a prime model. Moreover, every other countable model of T can be elementarily embedded into the saturated model (see Hodges' or Marker's textbooks on model theory [33, 34, 59]). We call a structure saturated (prime) if it is a saturated (prime) model of its own theory. An important model-theoretic property of prime and saturated models of a given theory T, in the case when they exist, is that they are unique up to isomorphism. We address this uniqueness property of the saturated models from a computability-theoretic point of view.

We have already provided examples of computably categorical prime models. To this list we can also add structures like the atomless Boolean algebra and the successor structure $(\mathbb{N}, S)$, where $S(n) = n + 1$. It is also not hard to give natural examples of prime but not computably categorical models. For instance, the natural numbers with their order $(\mathbb{N}, \leqslant)$ is the prime model but not computably categorical, since it has a computable presentation, where the successor relation is not computable. Another such example is $\mathcal{B}_\omega$, the Boolean algebra of finite and cofinite subsets of $\mathbb{N}$. In fact, all these structures have infinite computable dimension. There is also an example of a prime model of finite computable dimension $n > 1$ (see Chapter 4 for details).

However, despite the fact that countable saturated models (of a given theory) form one isomorphism type, all the known examples of computable saturated models are not computably categorical or their theories are $\aleph_0$-categorical. The latter means that they are prime models at the same time. A typical example here is the theory of vector spaces over the field of rational numbers. The non-saturated models of the theory are all the finite-dimensional vector spaces. They are all computably categorical. The countable saturated model is the countably infinite dimensional vector space. The saturated model is, however, not computably categorical. This is because there are two computable copies of the infinite dimensional vector space such that in one copy the dependency problem is computable and in the other it is not. Similarly, all the non-saturated models of the theory of one successor are computably categorical, while the saturated model of the theory is not. The reason for this is that the saturated model has two computable presentations such that in one the algebraic dependency relation is computable and in the other it is not.

Thus, it is interesting to give a nontrivial example of a computably categorical saturated model whose theory is not $\aleph_0$-categorical. Our main result in this chapter is the construction of such structure.

Theorem 3.2.1. *There exists a countable saturated not* $\aleph_0$*-categorical model that has a unique computable isomorphism type.*

The construction has two main parts. In the first part, we built a uniformly computably enumerable family of sets such that some specific enumerations of the family are equivalent via computable permutations. The construction of such family involves some facts and techniques from the theory of Kolmogorov complexity, which is widely used in the field of algorithmic randomness.

In the second part, we encode this family into a computably categorical structure using the construction of Fraïssé limits. The structure is a disjoint union of graphs, each of which encodes a certain set from the family. Namely, a set B is coded by a graph which is the Fraïssé limit of $\mathcal{K}(B)$, the class of all finite graphs which do not contain cycles of lengths in B. It turns out that the structure is computably categorical and saturated. Moreover, the theory of this structure has countably many countable models. Therefore, it is not $\aleph_0$-categorical.

We note that it is an open problem whether there exists a countable saturated model of finite computable dimension greater than 1.

The concepts of computable structure and computable categoricity can naturally be extended as follows. A structure is said to be a Σ^0_1-*structure* if it has universe ω and the open positive diagram of the structure, that is the set of all open formulas without negations true in the structure, forms a computably enumerable set of formulas. We stress that in the definition of a Σ^0_1-structure it is explicit that the domain of the structure is ω and the equality in the structure is a c.e. relation on ω and not the true equality of numbers. A Σ^0_1-structure is *computably categorical* if any two Σ^0_1-presentations of the structure are computably isomorphic. Clearly, every computable structure is also a Σ^0_1-structure. Therefore, if a computable structure is computably categorical when one considers Σ^0_1-presentations, then all Σ^0_1-presentations of the structure must be computable. Similarly, if a non-computable Σ^0_1-structure is computably categorical then the structure does not have a computable presentation. As the second application of our Kolmogorov complexity technique, we prove the following theorem.

Theorem 3.3.1. *There is an $\aleph_1$-categorical but not $\aleph_0$-categorical theory whose countable saturated model is a computably categorical Σ^0_1-structure.*

This partially answers the question of Goncharov about the existence of a computably categorical saturated model of an $\aleph_1$-categorical theory. However, the original question is still an open problem. We will also deal with Σ^0_1- and Π^0_1-structures in chapter 5.

In the last section of this chapter we will give a new proof of Theorem 2.1 from [47], which we restate here.

Theorem 1.7. *There exists an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T all models of which except the prime one have computable presentations.*

We point out here that in the proof of this theorem we use the same notion of A-cubes as in Chapter 2, where we constructed a theory T such that only the countable saturated model of T has a computable presentation.

The material of this chapter is published in Khoussainov/Semukhin/Stephan [49].

Chapter 4. Prime models of finite computable dimension

As we pointed out in the very beginning of the introduction, computable mathematics studies the effective versions of classical mathematical notions and constructions. One of the most fundamental notions in mathematics is that of an *isomorphism.* In algebra and model theory we usually identify isomorphic structures and consider them to be the same. However, when studying computable models, one can see that isomorphic structures might have different computability-theoretic properties. Therefore, we introduce the notion of a *computable isomorphism*, instead of the classical one, and use it as a tool to distinguish two different computable presentations of the same structure. This approach leads us to the notion of *computable dimension*, which is equal to the number of computable presentations of the structure up to computable isomorphism.

In this chapter we will answer the following open question in computable model theory. Does there exist a structure of computable dimension two which is the prime model of its own theory? It is easy to give examples of prime models with dimension 1 or ω. For instance, the countable dense linear order without endpoints and one successor structure $(\mathbb{N}, S)$ are computably categorical, while $(\mathbb{N}, \leqslant)$ and $\mathcal{B}_\omega$, the Boolean algebra of finite and cofinite subsets of the naturals, have infinite computable dimension.

However, it is much more difficult to construct a structure of finite computable dimension $k > 1$. Goncharov [21, 22] was the first to give an example of such structure. In [21] he constructed a uniformly computably enumerable (u.c.e.) family $\mathcal{F}$ of sets that has exactly two non-equivalent one-to-one computable enumerations. This family is then encoded into a computable graph $\mathcal{G}$ in such a way that the computable dimension of $\mathcal{G}$ is equal to the number of non-equivalent one-to-one computable enumerations of $\mathcal{F}$.

Since then many improvements to the construction have been made to obtain various strengthenings of this result. For example, Cholak/Goncharov/Khoussainov/Shore [6] showed that for each $k > 1$, there is a computably categorical structure $\mathcal{A}$ such that any expansion of $\mathcal{A}$ by a single constant has computable dimension k. This construction was further improved by Hirschfeldt/Khoussainov/Shore [31], who showed that it is possible to make the dimension of the expanded structure infinite.

The research on the structures of finite dimension is also related to the study of degree spectra of relations on computable models. The *degree spectrum* of a relation R on a computable structure $\mathcal{A}$ is the set of Turing degrees of images of R in all computable presentations of $\mathcal{A}$. Harizanov [26] showed that there exists a relation U in a structure $\mathcal{A}$ of computable dimension 2 such that $\mathrm{DgSp}_{\mathcal{A}}(U) = \{\mathbf{0}, \mathbf{d}\}$, where $\mathbf{d} \leqslant \mathbf{0}'$ and does not contain a c.e. set. Later on, Khoussainov and Shore [50] proved that there exists a relation U in a structure $\mathcal{A}$ of dimension 2 such that $\mathrm{DgSp}_{\mathcal{A}}(U) = \{\mathbf{0}, \mathbf{d}\}$, where $\mathbf{d}$ is c.e. but not computable. Hirschfeldt [29] further improved this result and showed that $\mathbf{d}$ can be any non-computable c.e. Turing degree.

All known examples of the structures of finite computable dimension $k > 1$ are not the prime models of their theories. Hence it was an open problem as to whether there exists a prime model of computable dimension 2. This question is especially interesting because prime models are relatively simple from a model-theoretic point of view. In fact, they are elementarily embeddable into any other model of their theory. So, the problem is whether it is possible to encode enough information into a prime model to construct such a structure of dimension two. The main result of this chapter is the construction of such structure.

Theorem 4.1.1. *There exists a computable structure of dimension two which is the prime model of its own theory.*

The construction is based on coding a u.c.e. family of sets $\mathcal{F}$ constructed by Goncharov [21] into a graph. The properties of the family that will be useful in our construction are listed below:

1) $\mathcal{F}$ has exactly two nonequivalent one-to-one computable enumerations;
2) every finite set $S \in \mathcal{F}$ contains an element $n(S)$, called a marker for a finite set, that does not belong to *any other* set from $\mathcal{F}$;
3) every infinite set $S \in \mathcal{F}$ contains an element $n(S)$, called a marker for an infinite set, that does not belong to *any other infinite* set from $\mathcal{F}$.

The last two properties of $\mathcal{F}$ allow us to make the coding in such a way that every element of the graph is definable by a first order formula. This implies that the structure is prime.

We also give examples of prime models of finite computable dimension in some specific classes of algebraic structures.

Theorem 1.8. *There are prime models of computable dimension 2 in the following classes of structures:*

1) undirected graphs (section 4.2.1),

2) partially ordered sets (section 4.2.2),

3) lattices (section 4.2.3),

4) integral domains expanded by finitely many constants (section 4.2.4).

The construction of these examples follows the technique from Hirschfeldt/Khoussainov/Shore/Slinko [32], where they developed the methods for coding directed graphs into undirected, irreflexive graphs, partial orders, lattices, integral domains, nilpotent groups, etc. These codings preserve the following computability-theoretic properties of the structures:

(a) degree spectra of the structures;

(b) degree spectra of relations on computable structures;

(c) computable dimensions of the structures as well as computable dimensions of their expansions by a single constant.

We will show that if in the original structure $\mathcal{A}$ every element was definable by a first order formula, then the structure $\mathcal{B}$, into which we encode $\mathcal{A}$, is prime. In fact, every $b \in \mathcal{B}$ is also definable by some formula or, in the case of integral domains, there is a formula with finitely many solutions that holds on b.

Chapter 5. Π_1^0-presentations of algebras

The study of computable models can naturally be extended to include a wider class of structures. This can be done by postulating that the atomic diagrams or natural fragments of the atomic diagrams are in some complexity class such as Σ_n^0 or Π_n^0. These classes of structures include computably enumerable (c.e.) algebras and co-c.e. algebras which we call Σ_1^0-algebras and Π_1^0-algebras, respectively. Roughly speaking, Σ_1^0-algebras are the ones whose positive atomic diagrams are computably enumerable, and Π_1^0-algebras are the ones whose negative atomic diagrams are computably enumerable. Typical examples are finitely presented algebras, like finitely presented groups or rings, and groups generated by finitely many computable permutations of ω.

There has been some research on Σ_1^0-algebras in the works by Feiner [13], Love [57], Kasymov [37, 38], Kasymov/Khoussainov [39] and Khoussainov/Lempp/Slaman [44]. However, not much is known about Π_1^0-algebras and their properties. The main goal of this chapter is to study the question as to which computable algebras are isomorphic to non-computable Π_1^0-algebras. Examples we have in mind are typical computable structures such as arithmetic $(\omega, S, +, \times)$, finitely generated term algebras and fields. We would like to know whether the isomorphism types of these typical computable structures contain non-computable but Π^0_1-algebras. In regard to this, it is worth noting that all these mentioned structures fail to be isomorphic to non-computable Σ^0_1-algebras. Hence, the existence of non-computable Π^0_1-presentations for such structures is of independent interest.

There are many algebraic structures that are naturally presented as Σ^0_1- or Π^0_1-algebras. For example, finitely presented groups and Lindenbaum algebras of c.e. first order theories are Σ^0_1-algebras. On the other hand, the group G generated by computable permutations $p_1, \ldots, p_n$ of $\mathbb{N}$ is a Π^0_1-algebra. If we consider a computable language $L \subseteq \Sigma^*$ in the alphabet $\Sigma = \{0, \ldots, k-1\}$, then the quotient structure $\mathcal{A} = (\Sigma^*, S_0, \ldots, S_{k-1})/ \sim_L$ is a Π^0_1-algebra, where $S_i(x) = xi$, and the congruence $\sim_L$ is defined as $x \sim_L y$ iff $\forall u \; (xu \in L \leftrightarrow yu \in L)$.

Non-computable presentations of Σ^0_1-algebras were studied by Khoussainov, Lempp, and Slaman in [44]. They showed that a given Σ^0_1-algebra $\mathcal{A}$ is non-computable if and only if $\mathcal{A}$ preserves all the facts true in $\mathcal{A}$. A *fact* here is a c.e. conjunction of the form $\&_{i\in\omega} \forall \bar{x} \neg \psi_i(\bar{x}, \bar{c})$, where each $\psi_i(\bar{x}, \bar{c})$ an atomic formula. We say that $\mathcal{A}$ *preserves the fact* φ if $\mathcal{A}$ and some proper homomorphic image of $\mathcal{A}$ satisfy φ.

This result implies that some well-known mathematical structures fail to possess non-computable Σ^0_1-presentations. Namely, the following structures do not possess non-computable Σ^0_1-presentations:

1) arithmetic $(\mathbb{N}, S, +, \times)$,
2) finitely generated term algebras (in fact, all finitely generated computable algebras),
3) computable fields.

The motivation for this work was to answer the question: do these structures possess non-computable Π^0_1-presentations? In this chapter we give a sufficient condition for a computable algebra to possess a non-computable Π^0_1-presentation. To formulate it, we introduce the notion of *term-separable* algebras (Definition 5.3.1). It turns out that the class of term-separable algebras is quite rich.

Proposition 5.3.3. *The following algebras are term-separable:*

1) arithmetic $(\mathbb{N}, S, +, \times)$*,*
2) any term algebra,
3) any infinite field,
4) any torsion-free abelian group,
5) any infinite vector space over a finite field.

Our main result of this chapter is the following theorem.

Theorem 5.4.1. *Let $\mathcal{A}$ be a computable term-separable algebra and* $\mathbf{d}$ *be any c.e. Turing degree. Then $\mathcal{A}$ possesses a Π^0_1-presentation of degree* $\mathbf{d}$*. In particular, it possesses a non-computable Π^0_1-presentation.*

From Proposition 5.3.3 and Theorem 5.4.1 we obtain the following corollary.

Corollary 5.4.8. *The following structures possess non-computable Π^0_1-presentations:*

1) *arithmetic* $(\mathbb{N}, S, +, \times)$,
2) *term algebras (both finitely and infinitely generated),*
3) *infinite computable fields,*
4) *computable torsion-free abelian groups,*
5) *infinite computable vector spaces over finite fields.*

Therefore, structures like arithmetic, finitely generated term algebras, and infinite computable fields possess non-computable Π^0_1-presentations but fail to possess non-computable Σ^0_1-presentations.

The material of this chapter is published in Khoussainov/Slaman/Semukhin [52].

Chapter 6. Finite automata presentable abelian groups

The notion of automatic or finite-automata (FA) presentable structure was first introduced and studied by Hodgson [35, 36] in mid 70's. About 20 years later this notion received attention again in the work by Khoussainov and Nerode [45], which gave rise to the modern development of the theory of automatic structures.

Roughly speaking, a structure $\mathcal{A}$ is called *automatic* or *FA presented* if, for some finite alphabet Σ, the domain of $\mathcal{A}$ is an FA recognizable subset of Σ^*. In addition to this, we require that the relations and the graphs of the operations of $\mathcal{A}$ are recognized by finite automata, which operate synchronously on their input. A structure is called *FA presentable* if it is isomorphic to an automatic structure. We will give the precise definitions in the main text of this chapter.

The computations performed by finite automata are much more restrictive than the ones by Turing machines. This implies that FA presentable structures have nice algorithmic properties. Probably the best-known fact about automatic structures is the following theorem proved independently by Hodgson and Khoussainov/Nerode [45].

Theorem 1.9. *The model checking problem for automatic structures is decidable, that is, given an FA presented structure $\mathcal{A}$, a formula $\varphi(\bar{x})$ and a tuple $\bar{a} \in |\mathcal{A}|$, there is an algorithm to decide whether $\mathcal{A} \models \varphi(\bar{a})$.*

In particular, this theorem implies that the first order theory of an FA presentable structure is decidable. Hodgson was the first who used this property to give a new proof of the decidability of Presburger arithmetic $\mathrm{Th}(\mathbb{N}, +)$. Another interesting fact about FA presentable structures is that they are closed under first order interpretations. Namely, if $\mathcal{A}$ is first order interpretable in $\mathcal{B}$, and $\mathcal{B}$ is FA presentable, then so is $\mathcal{A}$. Despite the restrictions imposed by finite automata, there are many natural examples of FA presentable structures, which makes them an interesting object of study both in mathematics and theoretical computer science.

There is a complete characterization of FA presentable structures in certain classes of algebraic structures such as Boolean algebras or well-ordered sets (ordinals). For example, an infinite Boolean algebra B is FA presentable if and only if $B \cong B_\omega^n$ for some n, where B_ω is the algebra of finite and cofinite subsets of natural numbers (Khoussainov/Nies/Rubin/Stephan [46]). An ordinal α is FA presentable if and only if $\alpha < \omega^\omega$ (Delhommé [9]).

However, not much is known about FA presentable structures in the other classes like groups, rings or linear orders. In [48] Khoussainov and Rubin posed the problem of characterizing automatic abelian groups (Problem 4). One of the very interesting open questions here is whether the group of rationals $(\mathbb{Q}, +)$ is FA presentable. If we consider torsion-free abelian groups, then the only known examples of automatic structures in this class were:

(a) the group of integers $(\mathbb{Z}, +)$;

(b) $R_p = \{m/p^k : \ m, k \in \mathbb{Z}, \ k \geqslant 0\}$, the group of rationals with denominators powers of p, where p is a product of different primes;

(c) finite direct products of these groups.

There was even a conjecture that these were all the examples of automatic structures in this class.

In this chapter we describe new FA presentable torsion-free abelian groups. To construct such examples, we use the method of amalgamated products. We will show that under certain conditions the amalgamated product of FA presentable groups is itself FA presentable.

As the first application of this method, we construct a new presentation of the group R_6, which contains FA recognizable subgroups isomorphic to R_2 and R_3 (see Example 6.4.3). Note that the standard presentation of R_6 described in Section 6.2 does not have this property.

The main application of this technique is the following theorem.

Theorem 1.10. *For every $n > 1$, there is a strongly indecomposable FA presentable torsion-free abelian group of rank n.*

Here a torsion-free abelian group A is *indecomposable* if for all B and C, $A = B \oplus C$ implies $B = \mathbf{0}$ or $C = \mathbf{0}$. A group is *strongly indecomposable* if it does not contain a decomposable subgroup of finite index. Therefore, the groups from the previous theorem are not isomorphic to the direct products of already known examples of automatic groups. Hence, they provide us new FA presentable structures in the class of torsion-free abelian groups.

Oliver and Thomas [68] discovered an interesting characterization of finitely generated FA presentable abelian groups. A group is called *abelian-by-finite* if it contains an abelian subgroup of finite index. The result of Oliver and Thomas states that if G is a finitely generated group, then G is FA presentable if and only if it is abelian-by-finite.

Nies and Thomas [66] recently showed that every finitely generated subgroup of an FA presentable group is abelian-by-finite. In light of this result, it is natural to ask whether the domain of every finitely generated subgroup in any FA presented group is FA recognizable. If the answer were positive, then the above mentioned result would be an easy corollary of the characterization by Oliver and Thomas. However, the answer is negative. As shown in Akiyama/Frougny/Sakarovitch [1], there is an automatic presentation of the group R_p where the set of integers is not FA recognizable. In the last section of this chapter we provide an even more interesting example of this type.

Theorem 1.11. *There is an FA presentation of the group $\mathbb{Z} \times \mathbb{Z}$, in which the domain of every nontrivial cyclic subgroup is not FA recognizable.*

Note that this contrasts with the standard presentation of $\mathbb{Z} \times \mathbb{Z}$, where every cyclic subgroup is FA recognizable.

The material of this chapter is published in Nies and Semukhin [65].

Chapter 2
A new spectrum of computable models

In this chapter we construct an uncountably categorical theory T such that the spectrum of computable models of T is equal to $\{\omega\}$. The chapter is organized as follows. The next section contains the proof of a computability-theoretic result that we will use in the construction of the desired theory. In Section 2.2 we introduce the basic building blocks of the models of this theory, which are called cubes. Finally, the last section contains the proof our main result. The necessary background and preliminaries are provided in the General Introduction.

2.1 A computability-theoretic result

Limitwise monotonic functions were introduced by N. G. Khisamiev [41, 42, 43] and have found a number of applications in computable model theory. In particular, Khoussainov, Nies, and Shore [47] used them to show that $(\omega+1)\setminus\{0\}$ is realized as a spectrum. We now introduce a related notion.

Let $[\omega]^{<\omega}$ denote the collection of all finite sets of natural numbers, and let ∞ be a special symbol. We define the class of S-limitwise monotonic functions from ω to $[\omega]^{<\omega}\cup\{\infty\}$, where S is an infinite set. This class captures the idea of a family $A_0, A_1, \ldots$ of uniformly c.e. sets, each of which is either finite or equal to S (represented by the symbol ∞), such that we can enumerate the set of i's for which $A_i = S$.

Definition 2.1.1. Let S be an infinite set of natural numbers. An *S-limitwise monotonic function* is a function $f : \omega \to [\omega]^{<\omega}\cup\{\infty\}$ for which there is a computable function $g : \omega\times\omega \to [\omega]^{<\omega}\cup\{\infty\}$ such that

1. $f(n) = \lim_s g(n,s)$ for all n, and
2. for all $n, s \in \omega$, the following properties hold:
    (a) if $g(n, s+1) \neq \infty$, then $g(n,s) \subseteq g(n,s+1)$,
    (b) if $g(n,s) = \infty$, then $g(n,s+1) = \infty$, and
    (c) if $g(n,s) \neq \infty$ and $g(n,s+1) = \infty$, then $g(n,s) \subset S$.

We refer to g as a *witness* to f being S-limitwise monotonic.

Note that if f is an S-limitwise monotonic function then its witness g can be chosen to be primitive recursive.

Definition 2.1.2. A collection of finite sets is *S-monotonically approximable* if it is equal to $\{f(n) : f(n) \neq \infty\}$ for some S-limitwise monotonic function f.

The main result of this section is the following computability-theoretic proposition which shows that there is an infinite set S and a family of sets that is not S-monotonically approximable and has certain properties that will allow us to code it into a model of an $\aleph_1$-categorical theory.

Proposition 2.1.3. *There exist an infinite c.e. set S and uniformly c.e. sets $A_0, A_1, \ldots$ with the following properties:*

1. *each A_i is either finite or equal to S,*
2. *if $x \in S$, then $x \in A_i$ for almost all i,*
3. *if $x \notin S$, then $x \in A_i$ for only finitely many i,*
4. *if A_i is finite, then there is a $k \in A_i$ such that $k \notin A_j$ for all $j \neq i$, and*
5. *$\{A_i : |A_i| < \omega\}$ is not S-monotonically approximable.*

Proof. Let $g_0, g_1, \ldots$ be an effective enumeration of all primitive recursive functions from $\omega \times \omega$ to $[\omega]^{<\omega} \cup \{\infty\}$ such that for all $n, s \in \omega$, if $g_e(n, s+1) \neq \infty$, then $g(n, s) \subseteq g(n, s+1)$, and if $g(n, s) = \infty$, then $g(n, s+1) = \infty$.

We want to build a set S and a family $A_0, A_1, \ldots$ to satisfy 1–3 and the requirements $\mathcal{R}_e$ stating that if g_e is a witness to some function f being S-limitwise monotonic, then $\{A_i : |A_i| < \omega\}$ is not S-monotonically approximable via f.

For each e, we define a procedure for enumerating A_e. We think of the procedures as alternating their steps, with the eth procedure taking place at stages of the form $\langle e, k \rangle$, which we call *e-stages*. All procedures may enumerate elements into S. The eth procedure is designed to satisfy $\mathcal{R}_e$ by ensuring that if g_e is a witness to some function f being S-limitwise monotonic and every $f(n) \neq \infty$ is equal to some A_i, then A_e is finite and not equal to $f(n)$ for any n. The eth procedure works as follows.

Let $A_e[s]$ and $S[s]$ denote the sets of all numbers enumerated into A_e and S, respectively, by the end of stage s.

The main idea is to find an appropriate number n_e such that if $\lim_s g_e(n, s) = A_e$ for some n, then $n = n_e$, and let $A_e[s]$ always contain an element not in $g_e(n_e, s)$, thus ensuring that either A_e is finite but $\lim_s g_e(n_e, s) \neq A_e$ or $g_e(n_e, s)$ is eternally playing catch-up and hence does not come to a limit.

At the first e-stage s, put $\langle e, 0 \rangle$, $\langle e, 1 \rangle$, and all elements of $S[s]$ into A_e. Let $m_{e,s} = 1$ and let n_e be undefined. (For each e-stage t, we will let $m_{e,t}$ be the largest m such that $\langle e, m \rangle \in A_e[t]$.)

At any other e-stage s, proceed as follows. Let t be the previous e-stage. If n_e is undefined and there is an $n \leqslant s$ such that $g_e(n, s) = A_e[t]$, then let $n_e = n$. If n_e is now defined and $g_e(n_e, s) = A_e[t]$, then put $\langle e, m_{e,t} - 1 \rangle$ into S, put $\langle e, m_{e,t} + 1 \rangle$ and all elements of $S[s]$ into A_e, and let $m_{e,s} = m_{e,t} + 1$. Otherwise, let $m_{e,s} = m_{e,t}$ and do nothing else.

This finishes the description of the eth procedure. Running all the procedures concurrently, as described above, we build a uniformly c.e. collection of sets

$A_0, A_1, \ldots$ and a c.e. set S. Now our goal is to show that these sets satisfy the properties in the statement of the proposition.

Since at every stage s at which we put numbers into A_e we put $S[s]$ into A_e and the second largest element of $A_e[s-1]$ into S, every infinite A_e is equal to S. This shows that the first property in the proposition holds.

Since, for each e, we put $S[s]$ into A_e, where s is the first e-stage, every element of S is in cofinitely many A_e's. This shows that the second property in the proposition holds.

Since the only way a number of the form $\langle e, k\rangle$ can enter A_i for $i \neq e$ is if it first enters S, every number that is in infinitely many A_i's must be in S. This shows that the third property in the proposition holds.

If A_e is finite, then $m = \lim_s m_{e,s}$ exists, and $\langle e, m\rangle$ is in A_e but not in A_j for $j \neq e$. This shows that the fourth property in the proposition holds.

We now show that the last property in the proposition holds. Assume for a contradiction that $\{A_i : |A_i| < \omega\} = \{f(n) : f(n) \neq \infty\}$ for some S-limitwise monotonic function f witnessed by g_e. Then n_e must eventually be defined, since, otherwise, A_e is finite but not in the range of f.

First, suppose that $f(n_e) \neq \infty$. At the e-stage s_0 at which n_e is defined, $g_e(n_e, s_0)$ contains $\langle e, 0\rangle$ and $\langle e, 1\rangle$. If there is no e-stage $s_1 > s_0$ at which $g_e(n_e, s_1) = A_e[s_0]$, then $f(n_e)$ cannot equal any of the A_i's since A_e is then the only one of our sets that contains $\langle e, 1\rangle$, and $\langle e, 1\rangle \in g_e(n_e, s_0)$. So, there must be such an e-stage s_1. Note that $g_e(n_e, s_1)$ contains $\langle e, 2\rangle$. By the same argument, there must be an e-stage $s_2 > s_1$ such that $g_e(n_e, s_2) = A_e[s_1]$, and this set contains $\langle e, 3\rangle$. Proceeding in this way, we see that $g_e(n_e, s)$ never reaches a limit.

Now suppose that $f(n_e) = \infty$. Let s_0 be the least s such that $g_e(n_e, s) = \infty$, and let t be the largest e-stage less than s_0. It is easy to check that $\langle e, m_{e,t} - 1\rangle \in g(n_e, t)$ but $\langle e, m_{e,t} - 1\rangle \notin S[t]$. We never put $\langle e, m_{e,t} - 1\rangle$ into S after stage t, so, in fact, $\langle e, m_{e,t} - 1\rangle \notin S$. Since $g_e(n_e, t) \subseteq g_e(n_e, s_0 - 1)$, we have $g_e(n_e, s_0 - 1) \not\subseteq S$, contradicting the choice of g_e.

□

2.2 Cubes

In this section we introduce a special family of structures which we call cubes. These will be used in the next section to build an $\aleph_1$-categorical theory. They generalize the n-cubes and ω-cubes used in [47].

We work in the language $\mathcal{L} = \{P_i : i \in \omega\}$, where each P_i is a binary predicate symbol. We will define structures for sublanguages $\mathcal{L}'$ of $\mathcal{L}$. Any such structure can be thought of as an $\mathcal{L}$-structure by interpreting the P_i's not contained in $\mathcal{L}'$ by the empty set. We denote the domain of a structure denoted by a calligraphic letter such as $\mathcal{A}$ by the corresponding roman letter A.

We begin with the following inductive definition of the finite cubes.

Definition 2.2.1. *Base case.* For $n \in \omega$, an *(n)-cube* is a structure $\mathcal{A} = (\{a, b\}; P_n^{\mathcal{A}})$, where $P_n^{\mathcal{A}}(x, y)$ holds if and only if $x \neq y$.

Inductive Step. Now suppose we have defined the σ-cubes for a non-repeating sequence $\sigma = (n_1, \ldots, n_k)$, and let $n_{k+1} \notin \sigma$. An *$(n_1, \ldots, n_k, n_{k+1})$-cube* is a structure $\mathcal{C}$ defined in the following way. Take two isomorphic σ-cubes $\mathcal{A}$ and $\mathcal{B}$ such that $A \cap B = \emptyset$ and let $f : \mathcal{A} \to \mathcal{B}$ be an isomorphism. Let $\mathcal{C}$ be the structure

$$(A \cup B; P_{n_1}^{\mathcal{A}} \cup P_{n_1}^{\mathcal{B}}, \ldots, P_{n_k}^{\mathcal{A}} \cup P_{n_k}^{\mathcal{B}}, P_{n_{k+1}}^{\mathcal{C}}),$$

where $P_{n_{k+1}}^{\mathcal{C}}(x, y)$ holds if and only if $f(x) = y$ or $f^{-1}(x) = y$.

Example 2.2.2. Let σ be a finite non-repeating sequence. Consider $A = \mathbb{Z}_2^{|\sigma|}$ as a vector space over $\mathbb{Z}_2$ with a basis $b_1, \ldots, b_{|\sigma|}$. If we define a structure $\mathcal{A}$ with the domain A by letting $P_{\sigma(i)}^{\mathcal{A}}(x, y)$ iff $x + b_i = y$, then $\mathcal{A}$ is a σ-cube.

The following property of the finite cubes, which is easily checked by induction, shows that we could have taken Example 2.2.2 as the definition of the σ-cubes.

Lemma 2.2.3. *Let σ be a finite non-repeating sequence. Then any two σ-cubes are isomorphic.*

Furthermore, we have the following stronger property.

Lemma 2.2.4. *If σ is a finite non-repeating sequence and τ is a permutation of σ, then every τ-cube is isomorphic to every σ-cube.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be a σ-cube and a τ-cube respectively. By Lemma 2.2.3, we can assume that $\mathcal{A}$ and $\mathcal{B}$ are constructed as in Example 2.2.2. Since τ is a permutation of σ, there is a bijection f such that $\sigma(i) = \tau(f(i))$. Let φ be the vector space isomorphism induced by taking b_i to $b_{f(i)}$. We then have

$$\begin{aligned} P_{\sigma(i)}^{\mathcal{A}}(x, y) \text{ iff } x + b_i = y \text{ iff } \varphi(x) + \varphi(b_i) = \varphi(y) \\ \text{iff } \varphi(x) + b_{f(i)} = \varphi(y) \text{ iff } P_{\tau(f(i))}^{\mathcal{B}}(\varphi(x), \varphi(y)) \text{ iff } P_{\sigma(i)}^{\mathcal{B}}(\varphi(x), \varphi(y)). \end{aligned}$$

Thus φ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

□

So, instead of a "σ-cube", where $\sigma = (n_1, \ldots, n_k)$, we can write an "$A$-cube", where $A = \{n_1, \ldots, n_k\}$. (This notation matches that of [47], if we make the usual set-theoretic identification of n with $\{0, \ldots, n-1\}$.)

We now define the infinite cubes.

Definition 2.2.5. Let $\alpha = (n_0, n_1, \ldots)$ be an infinite non-repeating sequence of natural numbers. An *α-cube* is a structure of the form $\bigcup_{i \in \omega} \mathcal{A}_i$ where each $\mathcal{A}_i$ is an $\{n_0, \ldots, n_i\}$-cube and $\mathcal{A}_i \subset \mathcal{A}_{i+1}$.

As with the finite sequences, the order of an infinite sequence α does not affect the isomorphism type of the corresponding infinite cubes. So, we can talk about S-cubes, where S is an infinite set. To show that this is the case, we will use the following fact, which is easy to check. Suppose that $A \subset B \subset C$ are finite, $\mathcal{Z}$ is a C-cube, and $\mathcal{X} \subset \mathcal{Z}$ is an A-cube. Then there exists a B-cube $\mathcal{Y}$ such that $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$.

Lemma 2.2.6. *If σ is an infinite non-repeating sequence and τ is a permutation of σ, then every τ-cube is isomorphic to every σ-cube.*

Proof. Let $\sigma = (m_0, m_1, \ldots)$ be an infinite non-repeating sequence, and let $\tau = (n_0, n_1, \ldots)$ be a permutation of σ. Let $s_i = \{m_0, \ldots, m_i\}$ and $t_i = \{n_0, \ldots, n_i\}$.

Let $\mathcal{A}$ be a σ-cube and let $\mathcal{B}$ be a τ-cube. Then $\mathcal{A} = \bigcup_{i\in\omega} \mathcal{A}_i$, where each $\mathcal{A}_i$ is an s_i-cube and $\mathcal{A}_i \subset \mathcal{A}_{i+1}$. Similarly, $\mathcal{B} = \bigcup_{i\in\omega} \mathcal{B}_i$, where each $\mathcal{B}_i$ is a t_i-cube and $\mathcal{B}_i \subset \mathcal{B}_{i+1}$.

We build a sequence of finite partial isomorphisms $\varphi_0 \subseteq \varphi_1 \subseteq \cdots$ such that $A_i \subseteq \operatorname{dom} \varphi_{2i+1}$ and $B_i \subseteq \operatorname{rng} \varphi_{2i+2}$. We begin with $\varphi_0 = \emptyset$.

Given φ_{2i}, let $k \geqslant i$ be such that $A_k \supseteq \operatorname{dom} \varphi_{2i}$, and let l be such that $B_l \supseteq \operatorname{rng} \varphi_{2i}$ and $s_k \subseteq t_l$. Then there is an s_k-cube $\mathcal{C} \subseteq \mathcal{B}_l$ such that $\operatorname{rng} \varphi_{2i} \subseteq C$. Extend φ_{2i} to an isomorphism $\varphi_{2i+1} : \mathcal{A}_k \to \mathcal{C}$.

Given φ_{2i+1}, proceed in an analogous fashion to define a finite partial isomorphism φ_{2i+2} including B_i in its range.

Now $\varphi = \bigcup_{i\in\omega} \varphi_i$ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

□

2.3 The Main Theorem

In this section we prove the main result of this chapter.

Theorem 2.3.1. *There exists an $\aleph_1$-categorical but not $\aleph_0$-categorical theory whose only computably presentable model is the countable saturated one.*

Proof. Let $\{A_i\}_{i\in\omega}$ and S be as in Proposition 2.1.3. Fix an enumeration of $\{A_i\}_{i\in\omega}$ such that at each stage exactly one element is enumerated into some A_i. (For instance, we can take the enumeration given in the proof of Proposition 2.1.3.) Construct a computable model $\mathcal{M}_\omega = \bigcup_{n\in\omega} \mathcal{M}_\omega^n$ as follows. Begin with $\mathcal{M}_\omega^n[0] = \emptyset$ for all n. At stage $s+1$, if $A_n[s+1] \neq A_n[s]$ then extend $\mathcal{M}_\omega^n[s]$ to an $A_n[s+1]$-cube using fresh large numbers.

It is clear that this procedure can be carried out effectively. So, $\mathcal{M}_\omega$ is computable. Furthermore, $\mathcal{M}_\omega$ is the disjoint union of one A_n-cube for each $n \in \omega$. In particular, every infinite cube in $\mathcal{M}_\omega$ is an S-cube.

Now let $T = \operatorname{Th}(\mathcal{M}_\omega)$ be the first-order theory of $\mathcal{M}_\omega$. We show that T is $\aleph_1$-categorical but not $\aleph_0$-categorical, $\mathcal{M}_\omega$ is countable saturated, and the only computably presentable model of T (up to isomorphism) is $\mathcal{M}_\omega$.

We begin by showing that T is $\aleph_1$-categorical. Since T includes sentences saying that, for each n and x, there is at most one y such that $P_n(x, y)$, we are free to use the functional notation and write $P_n(x) = y$ instead of $P_n(x, y)$. For $n \in S$, let $k(n)$ be the number of elements $x \in M_\omega$ for which $P_n^{\mathcal{M}_\omega}(x)$ is not defined. For $n \notin S$, let $k(n)$ be the number of elements $x \in M_\omega$ for which $P_n^{\mathcal{M}_\omega}(x)$ is defined. Note that $k(n)$ is finite for all n.

It is easy to see that $\mathcal{M}_\omega$ satisfies the following list of statements which can be written as an infinite set $\Sigma \subset T$ of first-order sentences:

1. For each n, the relation P_n is a partial one-to-one function and $P_n(x) = y \to P_n(y) = x$.
2. For all $n \neq m$ and all x, we have $P_n(x) \neq P_m(x)$ and $P_n(x) \neq x$.
3. For all $n \neq m$ and all x, if $P_n(x)$ and $P_mP_n(x)$ are defined, then $P_m(x)$ and $P_nP_m(x)$ are defined, and $P_nP_m(x) = P_mP_n(x)$.
4. For all k, all $n > n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$, and all x, we have $P_{n_1} \dots P_{n_k}(x) \neq P_n(x)$.
5. For each $n \in S$, there are exactly $k(n)$ many elements x for which $P_n(x)$ is not defined.
6. For each $n \notin S$, there are exactly $k(n)$ many elements x for which $P_n(x)$ is defined.
7. If A_i is finite and $m \in A_i$ is such that $m \notin A_j$ for all $j \neq i$, then there exists a finite A_i-cube $\mathcal{C}_i$ such that $\forall x$ ($P_m(x)$ is defined $\to x \in \mathcal{C}_i$). (Note that $m \notin S$ and $\mathcal{C}_i$ has $k(m)$ many elements, so, together with Statements 3 and 6, this statement implies that $\mathcal{C}_i$ is not contained in a larger cube.)

Remark 2.3.2. Note that Statements 1 and 3 imply the following: for all $n \neq m$ and all u, if $P_n(u)$ and $P_m(u)$ are defined then $P_mP_n(u)$ and $P_nP_m(u)$ are defined and equal. To prove this let $v = P_n(u)$. By Statement 1, this implies that $P_n(v) = u$. Since $P_mP_n(v) = P_m(u)$ is defined, by applying Statement 3 with $x = v$, we have that $P_m(v)$ and $P_nP_m(v)$ are defined, and $P_nP_m(v) = P_mP_n(v)$. If we let $w = P_m(v)$, then $P_mP_n(u) = w$. Since $P_n(w) = P_nP_m(v) = P_mP_n(v) = P_m(u)$, Statement 1 implies that $P_nP_m(u) = P_nP_n(w) = w$. Thus $P_mP_n(u) = P_nP_m(u)$.

Now suppose that $\mathcal{M}$ is a model of Σ. Let $A \subseteq \omega$ and $x \in M$. Using the statements above, it is easy to check that $\forall n \in A$ ($P_n^{\mathcal{M}}(x)$ is defined) if and only if x belongs to an A-cube. It is also clear that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are A-cubes in $\mathcal{M}$ and $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$, then $\mathcal{C}_1 = \mathcal{C}_2$.

It follows that $\mathcal{M}$ is the disjoint union of the components $\mathcal{M}_0$ and $\mathcal{M}_1$, where $\mathcal{M}_0$ is the disjoint union of exactly one A_i-cube for each finite A_i. Let $x \in M_1$. If $n \in S$, then there are $k(n)$ elements in M_0 on which $P_n^{\mathcal{M}}$ is not defined. Statement 5 says that there are exactly $k(n)$ such elements in M. Hence $P_n^{\mathcal{M}}(x)$ is defined. Similarly, Statement 6 implies that if $n \notin S$, then $P_n^{\mathcal{M}}(x)$ is not defined. Therefore, x belongs to an S-cube. Thus, $\mathcal{M}_1$ is a disjoint union of S-cubes.

Let $\mathfrak{C}$ be the class of all the structures that are the disjoint unions of exactly one A_i-cube for each finite A_i and some finite or infinite number of S-cubes. Clearly, any structure in $\mathfrak{C}$ is a model of Σ, and we have shown that any model of Σ is in $\mathfrak{C}$. Let $\mathcal{M}$ be a model of Σ. Each of the S-cubes in $\mathcal{M}$ is countable, so if $|M| = \aleph_1$, then there must be $\aleph_1$ many such S-cubes. Therefore, any two models of Σ of size $\aleph_1$ are isomorphic, and hence Σ is uncountably categorical. It now follows by the Łoś-Vaught Test that any model of Σ is a model of T. Thus T is uncountably categorical and, since $\mathfrak{C}$ contains infinitely many nonisomorphic countable structures, T is not countably categorical.

Lemma 2.3.3. *Let $\mathcal{M}$ be a computable model of T. Then $\mathcal{M}$ contains infinitely many S-cubes.*

Proof. Assume for a contradiction that $\mathcal{M}$ contains a finite number r of S-cubes (which may be 0). We can assume without loss of generality that the domain of $\mathcal{M}$ is ω. Let $\mathcal{M}_s$ be the structure obtained by restricting the domain of $\mathcal{M}$ to $\{0, \ldots, s\}$ and the language to $P_0, \ldots, P_s$. Choose one element from each S-cube, say $c_1, \ldots, c_r$. Define a computable function $g : \omega \times \omega \to [\omega]^{<\omega} \cup \{\infty\}$ as follows.

If $x > s$, then $g(x, s) = \emptyset$. If x is connected to some c_i in $\mathcal{M}_s$, then $g(x, s) = \infty$. Otherwise, $g(x, s)$ is the set of all $k \leqslant s$ for which there is a $y \leqslant s$ such that $P_k^{\mathcal{M}}(x, y)$.

Clearly, $g(x, s)$ is computable. Also, if x belongs to some A_i-cube in $\mathcal{M}$, then $g(x, s) \subseteq A_i$, and if $g(x, s) = \infty$, then x must belong to an S-cube. It is now easy to check that $f(x) = \lim_s g(x, s)$ is S-limitwise monotonic and $\{f(x) : f(x) \neq \infty\} = \{A_i : |A_i| < \omega\}$. But this contradicts the fact that $\{A_i : |A_i| < \omega\}$ is not S-monotonically approximable.

□

Since $\mathcal{M}_\omega$ is computable, it contains infinitely many S-cubes and, therefore, is countable saturated. The other countable models of T have only finitely many S-cubes and hence do not have computable presentations.

□

Chapter 3
Computably categorical saturated models

The main result of this chapter is the construction of a computably categorical saturated model whose theory is not $\aleph_0$-categorical. The introduction to this problem and the necessary background are presented in the General Introduction. The outline of the chapter is as follows.

In the first section we construct a specific uniformly computably enumerable family $\{B_x\}_{x\in\omega}$ of subsets of natural numbers. The definition of the family is based on the notion of Kolmogorov complexity. We also prove that some special enumerations of the family are equivalent to each other via computable permutations.

In Section 3.2 we provide an example of a countable saturated not $\aleph_0$-categorical model with exactly one computable isomorphism type. The idea is to code the family $\{B_x\}_{x\in\omega}$ from the previous section into a saturated structure so that the computable copies of the structure induce the special enumerations of the family. The construction of this structure is based on the well-known model-theoretic concept of Fraïssé limits.

In Section 3.3 of the chapter we also address a question of S. Goncharov that asks if there exists an $\aleph_1$-categorical but not $\aleph_0$-categorical countable saturated model that has a unique computable isomorphism type. We partially answer the question of Goncharov positively by providing a saturated computably categorical Σ^0_1-structure whose theory is $\aleph_1$- but not $\aleph_0$-categorical. Unfortunately, the structure is not computable. Our construction encodes the family $\{B_x\}_{x\in\omega}$.

In Section 3.4 we provide an alternative proof of the main result in [47]. There, an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T is constructed such that all models of T but the prime are computable. Our construction is again based on coding the family $\{B_x\}_{x\in\omega}$ into an $\aleph_1$-categorical but not $\aleph_0$-categorical theory.

3.1 The role of Kolmogorov Complexity

The first result is the construction of an auxiliary family of computably enumerable sets $B_0, B_1, \ldots$ with the following properties:

- every finite member of this family occurs only once;
- all infinite members are equal and occur infinitely often in every computable enumeration of the family.

The construction goes as follows. Let U be a universal partial computable function in the sense that for every further partial computable function ψ, there is a constant

c such that for all x in the domain of ψ, there is a $y \leqslant c(x+1)$ with $U(y) = \psi(x)$. Then the Kolmogorov complexity C (based on U) of any number z is defined as

$$C(z) = \min\{\log(x) : U(x)\downarrow = z\},$$

where the logarithm $\log(x)$ is defined as the smallest natural number y with $2^y \geqslant x$. The rationale behind this definition is that it should roughly invert exponentiation, have base 2, and avoid undefined places, proper fractions, and irrational numbers. Note that $C(z) \geqslant \log(z)$ for infinitely many z. The reader should consult standard textbooks [5, 56, 67, 71] for more information on Kolmogorov Complexity and Computability Theory. The family in question is now defined as follows.

Definition 3.1.1. Let $A = \{x : C(x) < \log(x)\}$ be the set of compressible or nonrandom numbers. Define

$$B_x = \begin{cases} \{x\} \cup \{y \in A : y < \log(x)\} & \text{if } x \notin A; \\ A & \text{if } x \in A. \end{cases}$$

The family $B_0, B_1, \ldots$ is uniformly computably enumerable as $\{x\} \cup \{y \in A : y < \log(x)\} \subseteq B_x$ for all x and the set A is computably enumerable. So, a uniform enumeration of the B_x's starts with an enumeration of $\{x\} \cup \{y \in A : y < \log(x)\}$ for each x and later adds all other elements of A in the case when x turns out to be an element of A.

Theorem 3.1.2. *If $E_0, E_1, \ldots$ is a computable enumeration of c.e. sets such that for every x, there is y with $E_x = B_y$, and for every $y \notin A$, there is a unique x with $B_y = E_x$, then there is a computable permutation f with $B_y = E_{f(y)}$ for all y.*

Proof. For every $y \notin A$, there is a unique x with $y \in E_x$. As every $y \in A$ satisfies $y \in E_x$ for almost all x, one has that there are infinitely many x with $y \in E_x$. Thus there is a computable function g such that $y \in E_{g(y)}$ for all y. This function can be obtained by searching in parallel in all $E_0, E_1, \ldots$ until an x with $y \in E_x$ is found.

If $y \notin A$, then $E_{g(y)} = B_y$ as B_y is the unique set in the enumeration $B_0, B_1, \ldots$ containing y. Thus $E_{g(y)}$ has to be equal to B_y. If $y \in A$ and $E_{g(y)} = A$, then $B_y = E_{g(y)}$. If $y \in A$ but $E_{g(y)} = B_x$ for some $x \notin A$, then $y < \log(x)$ and x is the unique element of $E_{g(y)}$ which is larger than $\log(x)$. Thus one can compute x from y and $\log(x)$. As the Cantor pairing function is invertible, one can also compute x from $(y + \log(x))(y + \log(x) + 1)/2 + \log(x)$. As $y < \log(x)$, the logarithm of this expression is roughly $2 \log \log(x)$. So, on the one hand, there is a constant c_1 with $C(x) \leqslant 2 \log \log(x) + c_1$. On the other hand, $\log(x) \leqslant C(x)$ as $x \notin A$. Therefore, $\log(x) \leqslant 2 \log \log(x) + c_1$ and hence there are only finitely many such x. Thus it follows that g satisfies $E_{g(y)} = B_y$ for all but finitely many y. Since the errors occur on finitely many $y \in A$, the other elements of A are mapped to E_x's with $E_x = A$. By modifying g at finitely many places, one obtains that $\forall y\, (E_{g(y)} = B_y)$. Note that A occurs in the enumeration $E_0, E_1, \ldots$ infinitely often as A is not computable and cannot be of the form $g^{-1}(D)$ for any finite set D.

Now let I be an infinite computable subset of A which includes all the y's satisfying the condition $g(y) \in \{g(z) : z < y\}$. Furthermore, let x be any index. If $E_x = B_y$ for a $y \notin A$, then one can compute y from x by inverting g, and hence $C(y) \leqslant C(x) + c_2$ for some constant c_2. As y is incompressible, $\log(y) \leqslant C(x) + c_2 \leqslant \log(x) + c_3$ for some constant c_3. Thus one knows that whenever $x \neq g(y)$ for all y with $\log(y) \leqslant \log(x) + c_3$, then $E_x = A$. So, the set of all x with $E_x = A$ is computably enumerable: $E_x = A$ iff either $g(y) = x$ for some $y \in A$ or $g(y) \neq x$ for finitely many y's with $\log(y) \leqslant \log(x) + c_3$. In particular, there is an infinite computable set J such that $E_x = A$ for all $x \in J$ and $J \cup g(\mathbb{N}) = \mathbb{N}$.

Now define $f(y)$ to be the n-th element of J whenever y is the n-th element of the computable set $I \cup g^{-1}(J)$, and let $f(y) = g(y)$ otherwise. Again, $E_{f(y)} = B_y$ for all y as f coincides with g on those y's for which B_y is finite, while f is modified from one index of A to another in the case when $y \in I \cup g^{-1}(J)$. Furthermore, by the construction, f is a permutation. It is also easy to see that f is computable.

□

3.2 The first application

Our main result in this section is the following theorem.

Theorem 3.2.1. *There exists a countable saturated not $\aleph_0$-categorical model that has a unique computable isomorphism type.*

We need to recall the construction of Fraïssé limits. Let $\mathcal{K}$ be a class of finite structures closed under isomorphisms. We assume that the language of the structures is finite and contains only relational symbols. Assume that the class $\mathcal{K}$ has the following properties:

1. Hereditary property (*HP*): for all $A \in \mathcal{K}$, if B is a substructure of A, then $B \in \mathcal{K}$.
2. Joint embedding property (*JEP*): for all $A, B \in \mathcal{K}$, there exists a $C \in \mathcal{K}$ such that A and B can be embedded into C.
3. Amalgamation property (*AP*): for all $A, B, C \in \mathcal{K}$, if $f\colon A \to C$ and $g\colon A \to B$ are embeddings, then there exists a structure $D \in \mathcal{K}$ and embeddings $h\colon B \to D$ and $k\colon C \to D$ such that $kf = hg$ on A.

A structure D is called *weakly homogeneous* if it has the property

> if A, B are finite substructures of D, $A \subseteq B$, and $f\colon A \to D$ is an embedding, then there is an embedding $g\colon B \to D$ which extends f.

A structure is called *ultrahomogeneous* if any finite partial isomorphism of the structure into itself can be extended to an automorphism. A finite or countable structure is ultrahomogeneous if and only if it is weakly homogeneous (see Lemma 7.1.4(b) in [33]).

The age of a structure D is the class of all finite structures embeddable in D. There is a well-known result in model theory that connects the ultrahomogeneous structures with the classes $\mathcal{K}$ that possess the properties *HP*, *JEP*, and *AP*. It is stated in the following theorem.

Theorem 3.2.2. *For any class $\mathcal{K}$ that has HP, JEP, and AP, there exists a unique at most countable ultrahomogeneous structure* $\lim(\mathcal{K})$ *whose age coincides with $\mathcal{K}$. Moreover, under our assumption on the language of $\mathcal{K}$, the structure* $\lim(\mathcal{K})$ *is $\aleph_0$-categorical.*

The structure $\lim(\mathcal{K})$ is called the *Fraïssé limit* of the class $\mathcal{K}$. We restate this theorem with an eye towards computable categoricity.

Theorem 3.2.3. *Let $\mathcal{K}$ be a class of finite structures with the following properties.*

1. *$\mathcal{K}$ has the properties HP, JEP, and AP.*
2. *There exists a sequence $H_0, H_1, \dots$ such that*
    - *$\{H_n \mid n \in \omega\} \subseteq \mathcal{K}$ and each $A \in \mathcal{K}$ is isomorphic to some H_n;*
    - *the domain and the atomic diagram of H_n is computable uniformly in n;*
    - *the function $n \mapsto |H_n|$ is computable.*

Then the Fraïssé limit of $\mathcal{K}$ is a computably categorical structure.

Proof. First, show that the Fraïssé limit of $\mathcal{K}$ is computable. Let $\{(A_i, B_i)\}_{i\in\omega}$ be a computable enumeration of pairs of structures from $\mathcal{K}$ such that $A_i \subseteq B_i$ and for every pair $A, B \in \mathcal{K}$ with $A \subseteq B$, there exist an i and an isomorphism $f\colon B \to B_i$ such that $f(A) = A_i$. We now construct a computable structure D as follows. Let $D_0 = H_0$. Suppose that D_k has been constructed. By applying the *AP* property the necessary number of times, one can show that there is an isomorphic copy H'_n of some $H_n \in \mathcal{K}$ such that $D_k \subseteq H'_n$, and for all $i \leqslant k$, if A_i can be embedded in D_k, then for every embedding $f\colon A_i \to D_k$, there is an embedding $g\colon B_i \to H'_n$ extending f. For every n, we can effectively check whether there is a copy of H_n satisfying the condition above. So, let D_{k+1} be an isomorphic copy of H_n with the minimal index n satisfying that condition.

Now consider a computable structure $D = \bigcup_{k<\omega} D_k$. Since each D_k is in $\mathcal{K}$ and $\mathcal{K}$ possesses the *HP* property, the age of D is included in $\mathcal{K}$. Suppose A is in $\mathcal{K}$; then, by the *JEP*, there are $B \in \mathcal{K}$ such that $D_0 \subseteq B$ and an embedding $h\colon A \to B$. Now let a pair (A_i, B_i) be such that there is an isomorphism $f\colon B_i \to B$ with $f(A_i) = D_0$. By the construction, the embedding $f \restriction A_i\colon A_i \to D_0$ extends to an embedding $g\colon B_i \to D$. Hence, both B and A are in the age of D. Therefore, the age of D is exactly $\mathcal{K}$.

Let $A \subseteq B$ be finite substructures of D and $h\colon A \to D$ be an embedding. Since $A, B \in \mathcal{K}$, there are a pair (A_i, B_i) and an isomorphism $f\colon B_i \to B$ with $f(A_i) = A$. Furthermore, there is $k \geqslant i$ such that $hf \upharpoonright A_i$ is an embedding of A_i into D_k. By the construction, $hf \upharpoonright A_i$ extends to an embedding $g\colon B_i \to D_{k+1}$. Now $gf^{-1}\colon B \to D$ is an embedding that extends h. This proves that D is weakly homogeneous and hence ultrahomogeneous. Therefore, D is the Fraïssé limit of the class $\mathcal{K}$.

We now show that D is computably categorical. Let D' be a computable structure isomorphic to D; then there is a computable chain $\{D'_k\}_{k<\omega}$ of finite structures such that $D' = \bigcup_{k<\omega} D'_k$. We construct a computable isomorphism from D to D' as follows. Let f_0 be an embedding of D_0 into D'. Suppose that a finite partial embedding f_n has been constructed. If $n = 2m$, then look for the smallest $k \geqslant m$ such that $\mathrm{Dom}(f_n) \subseteq D_k$. Since D' is weakly homogeneous, there is an embedding $g\colon D_k \to D'$ that extends f_n, and it can be found effectively. So, let $f_{n+1} = g$. If $n = 2m+1$, then look for the smallest $k \geqslant m$ such that $\mathrm{Im}(f_n) \subseteq D'_k$. Since D is weakly homogeneous, there is an embedding $g\colon D'_k \to D$ that extends f_n^{-1}. So, let $f_{n+1} = g^{-1}$. Thus $f = \bigcup\limits_{n<\omega} f_n\colon D \to D'$ is a computable isomorphism.

□

We now define special classes of finite structures that have the properties *HP*, *JEP*, and *AP*. A *cycle* of length $n \geqslant 3$ is the graph $C_n = (\{1, \ldots, n\}, E)$ with $E = \{(1,2), (2,1), (2,3), (3,2), \ldots, (n-1,n), (n,n-1), (n,1), (1,n)\}$. We say that a graph *contains* a cycle of length n if there exists an embedding from C_n into the graph.

Let Y be a non-empty subset of the natural numbers. Consider the following class of finite directed graphs:

$$\mathcal{K}(Y) = \{(V,E) \mid \text{ if } (V,E) \text{ contains a cycle of length } n+3, \text{ then } n \in Y\}.$$

Lemma 3.2.4. *The class $\mathcal{K}(Y)$ possesses the properties HP, JEP, and AP.*

Proof. It is easy to see that $\mathcal{K}(Y)$ satisfies the properties *HP* and *JEP*. We prove that $\mathcal{K}(Y)$ satisfies *AP*. Let A, B, C be graphs in $\mathcal{K}(Y)$ such that A is a subgraph of B and C and the domain of A is the intersection of the domains of B and C. Define the graph D as follows. The domain of D is the union of the domains of B and C. The graph D contains all the edges of the graphs B and C. In addition, D contains all the edges of the form (b, c), where $b \in B \setminus A$ and $c \in C \setminus A$. It is not hard to see that the graph D built in this way belongs to $\mathcal{K}(Y)$.

□

We construct the desired structure C_ω as follows. To do this, we use the family $\{B_x\}_{x\in\omega}$ from the previous section. For each B_x, consider the limit structure $\lim \mathcal{K}(B_x)$. One can construct a sequence

$$\lim \mathcal{K}(B_0),\ \lim \mathcal{K}(B_1),\ \lim \mathcal{K}(B_2), \ldots$$

of these structures so that the following properties hold:

1) the graphs in this sequence are all pairwise disjoint;
2) the union of the domains of these graphs is ω;
3) the sequence is uniformly computable meaning that the set $\{(n,m) \mid m \in \lim \mathcal{K}(B_n)\}$ is computable.

The signature of C_ω consists of two binary relational symbols R and S. The domain of C_ω is ω. The relation R is the union of all the edges of the graphs that appear in the sequence above. The relation S consists of all the pairs (n,m) such that n, m belong to the same graph $\lim \mathcal{K}(B_x)$ for some x. Clearly, S is a computable equivalence relation. Thus, the structure C_ω constructed is computable. Our goal now is to show that C_ω satisfies the theorem stated in the beginning of this section.

Lemma 3.2.5. *The structure C_ω is computably categorical.*

Proof. Let D be any computable structure isomorphic to C_ω. Since the equivalence relation S in D is computable, there is a computable sequence $\{x_i\}_{i\in\omega}$ which consists of exactly one representative for each S-equivalence-class. Let E_i be a set such that the substructure of D with the domain $[x_i]_S$, the S-equivalence-class of x_i, is isomorphic to $\lim \mathcal{K}(E_i)$. Using the fact that D is computable, one can show that the sequence $\{E_i\}_{i\in\omega}$ is uniformly computably enumerable. Furthermore, for every x, there is y with $E_x = B_y$, and for every $y \notin A$, there is a unique x with $B_y = E_x$. Thus, by Theorem 3.1.2, there is a computable permutation f such that $B_i = E_{f(i)}$ for all i. By Theorem 3.2.3, $\lim \mathcal{K}(B_i)$ is a computably categorical structure. Note that the construction of the computable isomorphism between $\lim \mathcal{K}(B_i)$ and $\lim \mathcal{K}(E_{f(i)})$ can be done uniformly in i. Therefore, D is computably isomorphic to C_ω.

$\square$

Let $T = \mathrm{Th}(C_\omega)$ be the first-order theory of C_ω. Our goal is to show that C_ω is the countable saturated model of T. This is proved in the Lemma 3.2.6 below, which also characterizes the isomorphism types of the countable models of T. Call an S-equivalence-class *non-standard* if the restriction of R to this class is isomorphic to the Fraïssé limit $\lim \mathcal{K}(A)$. Consider the subsequence

$$\lim \mathcal{K}(B_{n_0}),\ \lim \mathcal{K}(B_{n_1}),\ \lim \mathcal{K}(B_{n_2}), \ldots$$

of the sequence

$$\lim \mathcal{K}(B_0),\ \lim \mathcal{K}(B_1),\ \lim \mathcal{K}(B_2), \ldots,$$

where $n_0, n_1, \ldots$ is the list of all numbers outside of A in increasing order. Consider the substructure of C_ω restricted to the subsequence above and denote it C_0. Let C_n be the structure obtained by adjoining to C_0 exactly n many copies of non-standard S-equivalence-classes.

Lemma 3.2.6. *The theory T satisfies the following properties.*

1. *C_0 is the prime model of T.*
2. *The class of all countable models of T is $\{C_0, C_1, C_2, \ldots, C_\omega\}$.*
3. *C_ω is the countable saturated model of T.*

Proof. Let us write down the axioms of T. First, note that the fact that x and y lie in the same component $\lim \mathcal{K}(B_n)$ of C_ω can be expressed by a first-order formula. Indeed, let $x, y \in \lim \mathcal{K}(B_n)$ for some n. Suppose that there is no edge from x to y and from y to x. Let B be the substructure of $\lim \mathcal{K}(B_n)$ with domain $\{x, y\}$. Let D be a graph with domain $\{x, y, z\}$ that extends B and contains the additional edges (x, z), (z, y). Note that D is in $\mathcal{K}(B_n)$ since it does not contain any cycle. By the weak homogeneity of $\lim \mathcal{K}(B_n)$, there is an embedding of D into $\lim \mathcal{K}(B_n)$ which extends the identity embedding on B. Therefore, we can express the fact that x, y belong to the same $\lim \mathcal{K}(B_n)$ by the formula

$$\varphi(x, y) = R(x, y) \vee R(y, x) \vee \exists z (R(x, z) \wedge R(z, y)).$$

We use the notation $\{\bar{c}\}$ for the set consisting of the elements of the tuple $\bar{c}$. Let $\psi^n(x_0, \ldots, x_{n-1})$ be a formula such that for any graph B

$$B \models \psi^n(\bar{b}) \quad \text{if and only if} \quad \{\bar{b}\} \text{ is a cycle of length } n \text{ in } B.$$

For any graph B and n-tuple of distinct elements $\bar{b}$ such that $B = \{\bar{b}\}$, let $\psi_{B,\bar{b}}(x_0, \ldots, x_{n-1})$ be the conjunction of the formulas $R(x_i, x_j)$ or $\neg R(x_i, x_j)$ satisfied by $\bar{b}$ in B. Thus, for any graph D and a tuple $\bar{d} \in D$ of the same length as $\bar{b}$,

$$D \models \psi_{B,\bar{b}}(\bar{d}) \quad \text{iff} \quad \text{there is an isomorphism from } B \text{ to } \{\bar{d}\} \text{ which takes } \bar{b} \text{ to } \bar{d}.$$

Let $S_n(x)$ be the formula which says the S-equivalence-class of x contains a cycle of length $n + 3$, that is

$$S_n(x) = \exists \bar{y} \Big(\psi^{n+3}(\bar{y}) \wedge \bigwedge_{i \leqslant n-1} S(x, y_i) \Big).$$

We also use the abbreviation $\bar{x} \in S_n$ for the formula $\bigwedge_{i \leqslant n-1} S_n(x_i)$, and $\bar{x} \in [z]$ for the formula $\bigwedge_{i \leqslant n-1} S(x_i, z)$, where $\bar{x} = x_0, \ldots, x_{n-1}$. Let U be the following list of axioms.

(Ax^0) S is an equivalence relation.

(Ax^1) $S(x, y) \to R(x, y) \vee R(y, x) \vee \exists z (R(x, z) \wedge R(z, y))$.

For every n:

(Ax^2_n) $\neg S(x, y) \to \neg \exists x_0, \ldots, x_{n+1} \Big(x_0 = x \wedge x_{n+1} = y \wedge \bigwedge_{i \leqslant n} (R(x_i, x_{i+1}) \vee R(x_{i+1}, x_i)) \Big)$.

For every $n \notin A$:

$(\mathrm{A}x_n^3)$ $\quad \exists x\ (S_n(x) \wedge \forall y\ (S_n(y) \rightarrow S(x, y)))$.

For every $n \notin A$, every $B, D \in \mathcal{K}(B_n)$ and every tuple $\bar{b}d$ of distinct elements such that $B = \{\bar{b}\}$ and $D = \{\bar{b}d\}$:

$(\mathrm{A}x^4_{n,B,D,\bar{b}d})$ $\quad (\forall \bar{x} \in S_n)\ (\psi_{B,\bar{b}}(\bar{x}) \rightarrow (\exists y \in S_n)\ \psi_{D,\bar{b}d}(\bar{x}, y))$.

If $\bar{b}$ is empty, then this sentence reduces to $(\exists y \in S_n)\ \psi_{D,d}(y)$.

For every $n \notin A$:

$(\mathrm{A}x_n^5)$ $\quad (\forall \bar{x} \in S_n) \bigvee_{B,\bar{b}} \psi_{B,\bar{b}}(\bar{x})$,

where the disjunction ranges over all pairs $B, \bar{b}$ such that $B \in \mathcal{K}(B_n)$ and $\bar{b}$ is a tuple of the same length as $\bar{x}$ with $B = \{\bar{b}\}$. Note that this disjunction is finite.

For every $B, D \in \mathcal{K}(A)$ and every tuple $\bar{b}d$ of distinct elements such that $B = \{\bar{b}\}$ and $D = \{\bar{b}d\}$:

$$(\mathrm{A}x^6_{B,D,\bar{b}d}) \quad \forall z \Big(\Big(\bigwedge_{i \leqslant k-1} \neg S_{n_i}(z) \Big) \rightarrow (\forall \bar{x} \in [z])\ (\psi_{B,\bar{b}}(\bar{x}) \rightarrow (\exists y \in [z])\ \psi_{D,\bar{b}d}(\bar{x}, y)) \Big),$$

where $n_0, \ldots, n_{k-1}$ are the indices of all the components $\lim \mathcal{K}(B_n)$ of C_ω into which D can not be embedded. Note that $n_i \notin A$ for all $i \leqslant k-1$. If $\bar{b}$ is empty, then this sentence reduces to

$$\forall z \Big(\Big(\bigwedge_{i \leqslant k-1} \neg S_{n_i}(z) \Big) \rightarrow (\exists y \in [z])\ \psi_{D,d}(y) \Big).$$

Let M be a countable model of U. The axioms $\mathrm{A}x^0$, $\mathrm{A}x^1$, and $\mathrm{A}x^2_n$ imply that S is an equivalence relation and that every S-equivalence-class is a connected component of M. For every $n \notin A$, $\mathrm{A}x^3_n$ states that there is a unique component that contains a cycle of length $n+3$. Denote this component by M_n.

When $\bar{b}$ is empty, $\mathrm{A}x^4_n$ says that every one-element structure in $\mathcal{K}(B_n)$ is embeddable in M_n. In general, $\mathrm{A}x^4_n$ says that

> if B, D are finite structures in $\mathcal{K}(B_n)$, D comes from B by adding one more element, and $f\colon B \rightarrow M_n$ is an embedding, then there is an embedding $g\colon D \rightarrow M_n$ which extends f.

Using induction on the number of elements, it is not hard to see that every structure in $\mathcal{K}(B_n)$ is embeddable in M_n. On the other hand, $\mathrm{A}x^5_n$ implies that any finite substructure of M_n is in $\mathcal{K}(B_n)$. Thus, the age of M_n is exactly $\mathcal{K}(B_n)$.

Using $\mathrm{A}x^4_n$ again and the induction on the size of $D \setminus B$, we can show that

> if $B, D \in \mathcal{K}(B_n)$, $B \subseteq D$, and $f\colon B \rightarrow M_n$ is an embedding, then there is an embedding $g\colon D \rightarrow M_n$ which extends f.

Thus M_n is a weakly homogeneous (and hence ultrahomogeneous) model of the age $\mathcal{K}(B_n)$. Therefore, M_n is isomorphic to $\lim \mathcal{K}(B_n)$. Note that, in particular, it means that $M_n \neq M_k$ whenever $n \neq k$.

Let M^* be a connected component of M that is different from all the M_n's. As shown above, any cycle of length $n+3$, for $n \notin A$, can appear only in M_n. So, the age of M^* is included in $\mathcal{K}(A)$. Let D be a one-element structure in $\mathcal{K}(A)$; then $\mathrm{A}x^6$ implies that D is embeddable in M^*. Now let $B, D \in \mathcal{K}(A)$, D comes from B by adding one more element, and $f\colon B \to M^*$ is an embedding. Suppose that D is not embeddable in $\lim \mathcal{K}(B_{n_0}), \ldots, \lim \mathcal{K}(B_{n_{k-1}})$, where $n_i \notin A$ for all $i \leqslant k-1$. In this case $\mathrm{A}x^6$ states that the embedding of B into any component of M other than $M_{n_0}, \ldots, M_{n_{k-1}}$ can be extended to an embedding of D into the same component. In particular, f can be extended to an embedding $g\colon D \to M^*$. Now, using induction on the size of D, it is not hard to show that every $D \in \mathcal{K}(A)$ is embeddable in M^*. Therefore, the age of M^* is exactly $\mathcal{K}(A)$.

Again, induction on the size of $D \setminus B$ tells us that if $B, D \in \mathcal{K}(A)$, $B \subseteq D$, and $f\colon B \to M^*$ is an embedding, then there is an embedding $g\colon D \to M^*$ which extends f. Thus M^* is a weakly homogeneous structure of the age $\mathcal{K}(A)$ and, therefore, is isomorphic to $\lim \mathcal{K}(A)$.

So, any countable model of U consists of exactly one component isomorphic to $\lim \mathcal{K}(B_n)$, for $n \notin A$, and a finite or infinite number of components isomorphic to $\lim \mathcal{K}(A)$. In other words, the class of all countable models of U is $\{C_0, C_1, C_2, \ldots, C_\omega\}$.

We now show that for every $i \in \omega$, C_i is elementarily equivalent to C_ω. To do this, we will use the method of Ehrenfeucht-Fraïssé games.

Definition 3.2.7. Let A and B be structures of the same language L and let γ be an ordinal. Then $\mathrm{EF}_\gamma[A, B]$, the *unnested Ehrenfeucht-Fraïssé game of length γ on A and B*, is defined as follows. There are two players $\forall$ and $\exists$. The game is played in γ steps. At the ith step of the play, player $\forall$ takes one of the structures A, B and chooses an element of this structure; then player $\exists$ chooses an element of the other structure. Each player is allowed to see and remember all previous moves in the play. At the end of the play, the sequences $\bar{a} = (a_i : i < \gamma) \in A$ and $\bar{b} = (b_i : i < \gamma) \in B$ have been chosen. The pair $(\bar{a}, \bar{b})$ is known as the *play*. We say that player $\exists$ *wins the play* $(\bar{a}, \bar{b})$ iff

for every unnested atomic formula φ of L, $A \models \varphi(\bar{a}) \Leftrightarrow B \models \varphi(\bar{b})$.

Note that if the language L contains no function symbols or constants, as in our case, then every formula of L is unnested. We write $A \approx_\gamma B$ to mean that player $\exists$ has a winning strategy for the game $\mathrm{EF}_\gamma[A, B]$.

Theorem 3.2.8. *Let L be a finite first-order language. Then for any two L-structures A and B, the following are equivalent.*

(I) $A \equiv B$.

(II) *For every $k < \omega$, $A \approx_k B$.*

For the proof and more details see chapters 3.2 and 3.3 in Hodges [33]. Now, let us fix any i and k and show that player $\exists$ has a winning strategy for the game $\mathrm{EF}_k[C_i, C_\omega]$.

The strategy for player $\exists$ that is described below has the following property. At the beginning of every step s, the sequences $\bar{a}_{s-1} = (a_0, \dots, a_{s-1})$, $\bar{b}_{s-1} = (b_0, \dots, b_{s-1})$ have been chosen by the players $\forall$ and $\exists$ such that the substructures $\{\bar{a}_{s-1}\}$ and $\{\bar{b}_{s-1}\}$ of C_i and C_ω, respectively, are isomorphic via the isomorphism that maps $\bar{a}_{s-1}$ to $\bar{b}_{s-1}$. Moreover, for every $t < s$, if $a_t \in \lim \mathcal{K}(B_n)$ and $b_t \in \lim \mathcal{K}(B_m)$, then either

(a) $n \notin A$, $\log(n) \leqslant k-3$ and $n = m$, or

(b) $n \notin A \Rightarrow k-3 < \log(n)$ and $m \notin A \Rightarrow k-3 < \log(m)$.

In both cases, we have that $B_n \cap [0, k-3] = B_m \cap [0, k-3]$, and hence any substructure of $\lim \mathcal{K}(B_n)$ with at most k elements is embeddable into $\lim \mathcal{K}(B_m)$, and vice versa.

Now suppose that at step s player $\forall$ has chosen $a_s \in C_i$. Then player $\exists$ chooses $b_s \in C_\omega$ such that the substructures $\{\bar{a}_s\}$ and $\{\bar{b}_s\}$ are isomorphic via the isomorphism that maps $\bar{a}_s$ to $\bar{b}_s$. Also, if a_s lies in the same component as a_t for some $t < s$, then b_s has to be in the same component as b_t. The fact that $|\bar{a}_s| \leqslant k$ ensures that player $\exists$ can always find such b_s. If the component of a_s does not contain any a_t for $t < s$, then player $\exists$ chooses b_s such that b_s is not in a component containing any b_t, for $t < s$, and that either the property (a) or (b) given above holds for the pair (a_s, b_s). Obviously, player $\exists$ can always find such a component because there are infinitely many components in both C_i and C_ω which are isomorphic either to $\lim \mathcal{K}(A)$ or to $\lim \mathcal{K}(B_n)$ for $n \notin A$ and $k < \log(n)$.

The case when player $\forall$ has chosen $b_s \in C_\omega$ is similar to the above.

It is not hard to see that this strategy is indeed a winning strategy for player $\exists$. Therefore, all the structures $C_0, C_1, \dots, C_\omega$ are elementary equivalent to each other, and U is the set of axioms for $T = \mathrm{Th}(C_\omega)$. Since T has countably many countable models, T has a countable saturated model and a prime model (see e.g. Corollary 4.3.8 in Marker [59]). None of the $\{C_i : i < \omega\}$ can be the saturated model because C_ω is not embeddable into any C_i. So, we can conclude that C_ω is in fact the countable saturated model of T.

□

3.3 The second application

In this section we partially answer the question of Goncharov about the existence of an $\aleph_1$-categorical but not $\aleph_0$-categorical saturated structure with a unique computable isomorphism type. Our answer is affirmative if one considers Σ^0_1-structures rather than computable structures. Here is the result.

Theorem 3.3.1. *There is a $\aleph_1$-categorical but not $\aleph_0$-categorical theory whose countable saturated model is a computably categorical Σ^0_1-structure.*

Proof. We use the family $\{B_x\}_{x\in\omega}$ constructed in Section 3.1. The language of the desired structure is given by the family $P_0, P_1, \dots$ of unary predicates. Define the structure M as follows. The domain of the structure is the set of natural numbers. For each x and y, $P_y(x)$ holds if and only if $y \in B_x$. Obviously, the structure is Σ^0_1.

Let T be the first-order theory of M. It can be described by the following set of axioms.

For every $i \notin A$:

($\mathrm{A}x^1_i$) There is a unique z such that $P_i(z)$.

For $i \notin A$, let c_i be a new constant interpreted as an element on which P_i holds. The axioms $\mathrm{A}x^1_i$ imply that these constants are definable in the original language.

For every $i \in A$:

($\mathrm{A}x^2_i$) $$\bigwedge_{\{j:\ i\notin B_j\}} \neg P_i(c_j) \ \wedge\ \forall z\Big(\Big(\bigwedge_{\{j:\ i\notin B_j\}} z \neq c_j\Big) \to P_i(z)\Big),$$

that is, P_i holds almost everywhere and does not hold only on the constants c_j for $i \notin B_j$. Note that if $i \in A$, then the set $\{j : i \notin B_j\}$ is finite; thus the conjunctions in $\mathrm{A}x^2_i$ are finite, and this is a first-order formula.

The theory T is not $\aleph_0$-categorical since the prime model is given by the substructure of M with the domain $\mathbb{N} - A$. The theory T is $\aleph_1$-categorical since any model of cardinality $\kappa > \aleph_0$ consists of the following elements:

- one element x with $\{n \mid P_n(x)\} = B$, for every finite set B in the enumeration $\{B_y\}_{y\in\omega}$;
- κ many elements x with $\{n \mid P_n(x)\} = A$.

The statement about Σ^0_1-categoricity can be proved by considering any further Σ^0_1-model M' with the domain $\mathbb{N}$ that is isomorphic to M; such a model defines a computable enumeration $E_0, E_1, \dots$ with $n \in E_y \Leftrightarrow P_n(y)$. It is easy to see that every B_x with $x \notin A$ appears in this enumeration only once, and that every E_y equals some B_x. By Theorem 3.1.2, there exists a computable permutation f such that $E_{f(x)} = B_x$ for all x. This computable permutation clearly induces a computable isomorphism between two Σ^0_1-structures M and M'.

□

3.4 The third application

Khoussainov, Nies, and Shore [47] gave an example of an $\aleph_1$-categorical but not $\aleph_0$-categorical theory such that all models of the theory except the prime one are computable. In this section we provide an alternative proof of this result using the family $\{B_x\}$ constructed in the Section 3.1.

To code this family, we generalize the notion of "cubes" introduced in [47]. Fix a language L consisting of binary relation symbols F_n (for $n \in \omega$), which we assume to be symmetric and irreflexive relations coding the edges of a hypercube. Each such cube is constructed from a given c.e. set X of parameters and has dimension $|X|$. For each element n of the set X, the edges along one dimension are realized by the relation F_n. More formally, this is done as follows.

For any subset X of ω, let the domain of the default presentation of an X-cube be the set $\{\sum_{m\in Y} 2^m : Y \subseteq X \wedge Y \text{ is finite}\}$ with $\sum_{m\in\varnothing} 2^m = 0$. On this domain, define the relation $F_n(x, y)$ to be true iff there is a finite subset $Y \subseteq X - \{n\}$ such that

$$\{x, y\} = \{\textstyle\sum_{m\in Y} 2^m, \sum_{m\in Y\cup\{n\}} 2^m\}.$$

An X-cube is then a structure isomorphic to the default presentation that we have just defined. Note that every default presentation of an X-cube is uniformly Σ^0_1 in X.

For example, a $\{0, 1, 3\}$-cube is an isomorphic copy of $\{0, 1, 2, 3, 8, 9, 10, 11\}$ together with the relations $F_0(0, 1)$, $F_0(2, 3)$, $F_0(8, 9)$, $F_0(10, 11)$, $F_1(0, 2)$, $F_1(1, 3)$, $F_1(8, 10)$, $F_1(9, 11)$, $F_3(0, 8)$, $F_3(1, 9)$, $F_3(2, 10)$, $F_3(3, 11)$. The other relations do not hold between the members of the $\{0, 1, 3\}$-cube.

Alternatively, an X-cube can be defined as follows (see [30] or Example 2.2.2). Consider $A = \mathbb{Z}_2^{|X|}$ as a vector space over $\mathbb{Z}_2$ with a basis $\{a_i\}_{i<|X|}$. Let $f : X \to |X|$ be a bijection. If for every $n \in X$ and every $x, y \in A$, we define $F_n(x, y) \Leftrightarrow x + a_{f(n)} = y$, then A will be an X-cube.

Definition 3.4.1. Let the set A and the sequence $B_0, B_1, \ldots$ be as in Definition 3.1.1. Let C_0 be the disjoint union of all those B_x-cubes with $x \notin A$. Furthermore, let C_n be the disjoint union of C_0 and n many A-cubes for $n \in \{1, 2, 3, \ldots, \omega\}$.

Note that C_ω is isomorphic to the disjoint union over all B_x-cubes for $x \in \omega$.

Proposition 3.4.2. *The structures $C_0, C_1, C_2, \ldots, C_\omega$ have all the same theory T, which is $\aleph_1$-categorical. C_0 is its prime model, and C_ω is its countable saturated model. The models $C_0, C_1, C_2, \ldots, C_\omega$ are the only countable models of T up to isomorphism.*

Here is a sketch of a proof. Let us consider the theory $T = \mathrm{Th}(C_\omega)$. We need formulate the list U of its axioms. One group of the axioms should say that the models of T consist of cubes. Furthermore, for every $x \in A$, we need an axiom saying that F_x is adjacent to all but n_x nodes, where n_x is the number of nodes in C_ω that are not adjacent to F_x. Similarly, for every $x \notin A$, we need an axiom saying that F_x is adjacent only to n_x many nodes, where n_x is the number of nodes in C_ω that are adjacent to F_x. Now one can show that $C_0, C_1, C_2, \ldots, C_\omega$ are all the countable models of U and all models of U of the same uncountable cardinality are isomorphic. This implies that the theory determined by U is complete and U is indeed the set of axioms for T. More details can be found in the proof of Theorem 2.3.1 in Chapter 2.

Theorem 3.4.3. *The models $C_1, C_2, \ldots, C_\omega$ have computable presentations, but C_0 does not have a computable presentation.*

Proof. This is clear for C_ω as there is a computable one-to-one enumeration (a_0, b_0), $(a_1, b_1), \ldots$ of pairs such that for each x, the set $\{a : \exists s\ [a_s = a \wedge b_s = x]\}$ is the domain of the default presentation of the B_x-cube from above. Then one defines that $F_n(s,t)$ holds iff $b_s = b_t$ and $F_n(a_s, a_t)$ in the default presentation of the B_{b_s}-cube. It is easy to see that the resulting model is computable and isomorphic to C_ω.

We now describe how to construct a computable presentation for C_1. Fix some $x_0 \in A$ and start the construction by enumerating all B_x-cubes in some effective way. Also start enumerating the set A. When at some stage s a number x is enumerated into A_s, we expand the finite part of the B_{x_0}-cube constructed so far and merge it with the finite part of the B_x-cube. To do this, we might need to use new edges F_t with $t \in A$ such that up to the stage s it has not been decided for any two nodes a, b whether $F_t(a, b)$ holds or not. So, we keep on enumerating A until we have enough of such edges. To build a computable presentation for C_n with $2 \leqslant n < \omega$, one need to add $n-1$ computable copies of the A-cube to the computable presentation of C_1.

Now assume for the contradiction that the prime model C_0 also has a computable presentation. Then there is a computable function mapping every n to a triple $(a(n), b(n), y(n))$ such that $2^n < y(n) \wedge F_{y(n)}(a(n), b(n))$. This function is total as there are infinitely many $x > 2^n$ such that $x \notin A$ and a copy of B_x-cube is merged into C_0. Let $x(n)$ denote the index of the $B_{x(n)}$-cube to which $a(n)$ belongs. Note that $B_{x(n)}$ is finite as C_0 is the prime model. Note that $x(n) \notin A$, $x(n) \geqslant y(n)$, and $C(x(n)) \geqslant \log(x(n))$. Now $x(n)$ can be computed from n and $\log(x(n))$ as $x(n)$ is the only number z larger than $\log(x(n))$ for which there is d with $F_z(a(n), d)$. So, $x(n)$ can be found by an exhausting search once $\log(x(n))$ and n are given. As $n \leqslant \log(x(n))$, we have that

$$C(x(n)) \leqslant C(\log(x(n)), n) + c \leqslant 2 \log\log(x(n)) + c'$$

for all n and some constants c, c'. Combining these two conditions, one has that

$$\log(x(n)) \leqslant 2 \log\log(x(n)) + c' \text{ for all } n.$$

But this is impossible since $x(n) > 2^n$ for all n. Therefore, C_0 cannot have a computable presentation.

□

Chapter 4
Prime models of finite computable dimension

In this chapter we construct an example of a prime model of computable dimension two. We also give examples of such models in some typical classes of algebraic structures. The background for this chapter and necessary preliminaries are provided in the General Introduction. The outline of the chapter is as follows.

In Section 4.1 we describe the construction of a directed graph of computable dimension two which is the prime model of its own theory. Then in Section 4.2, using the methods from Hirschfeldt/Khoussainov/Shore/Slinko [32], we will encode this graph into an undirected graph, a partial order, a lattice, and an integral domain. It follows from [32] that all these codings preserve computable dimensions. We show that these codings also preserve the property of being the prime model, except for the case of the integral domain, where we need to add finitely many constants to the language of the structure. In some cases we provide an explicit proof that a given structure has computable dimension two rather then referring the reader to [32].

4.1 The Main Construction

The main result of this chapter is the following theorem.

Theorem 4.1.1. *There exists a computable structure G of computable dimension two which is the prime model of its own theory.*

The structure G will be a directed graph. The proof is based on coding the u.c.e. family $\mathcal{F}$ constructed by Goncharov [21] into a computable graph of dimension two in such a way that every element of G can be defined by a first-order formula without parameters. This implies that G is the prime model of its theory. We now restate the result of S. Goncharov in more detail.

Definition 4.1.2. Let $\mathcal{F}$ be a u.c.e. family of sets. A *computable enumeration* $\mu : \omega \to \mathcal{F}$ is a mapping from ω onto $\mathcal{F}$ such that the set $\{(n, k) : n \in \mu(k)\}$ is c.e. We will also use the notation $\{A_i\}_{i\in\omega}$ for an enumeration μ, where $A_i = \mu(i)$.

An enumeration μ is *reducible* to ν, denoted $\mu \leqslant \nu$, if there is a computable function f such that $\mu(i) = \nu(f(i))$ for every $i \in \omega$. Two enumerations μ and ν are *equivalent*, denoted $\mu \equiv \nu$, if $\mu \leqslant \nu$ and $\nu \leqslant \mu$.

Theorem 4.1.3 (Goncharov [21])**.** *There exists a u.c.e. family $\mathcal{F}$ that has exactly two nonequivalent one-to-one computable enumerations. Moreover, $\mathcal{F}$ has the following properties:*

(i) *If $S \in \mathcal{F}$ is a finite set, then S contains an element $n(S)$, called a marker for a finite set S, that does not belong to* **any other** *set from $\mathcal{F}$.*

(ii) *If $S \in \mathcal{F}$ is an infinite set, then S contains an element $n(S)$, called a marker for an infinite set S, that does not belong to* **any other infinite** *set from $\mathcal{F}$.*

Remark 4.1.4. We may assume that the family $\mathcal{F}$ contains infinitely many one-element sets. Indeed, we can always take $\mathcal{F}' = \{2S : S \in \mathcal{F}\} \cup \{\{2k+1\} : k \in \omega\}$ instead of $\mathcal{F}$. The family $\mathcal{F}'$ has exactly two nonequivalent one-to-one computable enumerations. This follows from the fact that the index set of the subfamily $\{\{2k+1\} : k \in \omega\}$ is computable in any one-to-one computable enumeration of $\mathcal{F}'$.

Let $\{A_n^0\}_{n\in\omega}$ and $\{A_n^1\}_{n\in\omega}$ be two nonequivalent one-to-one computable enumerations of $\mathcal{F}$. For each $i = 0, 1$, fix a computable enumeration of $\{A_n^i\}_{n\in\omega}$ such that at every step s, exactly one element enters one of the A_n^i's.

We build two computable presentations G_0 and G_1 of the directed graph G using a step-by-step construction. Let G_i^s be the finite part of G_i constructed by the end of step s. When we add a new element to G_i^s, we always choose the least element that has not been used so far. At every step s, we will have that $G_i^s \subseteq G_i^{s+1}$ and $G_i = \bigcup_{s\in\omega} G_i^s$. We will use the following notations in our construction.

Definition 4.1.5. Let $n \in \omega$; the directed graph $[n]$ has $n+3$ many nodes $x_0, x_1, \ldots, x_{n+2}$ with an edge from x_0 to itself, an edge from x_{n+2} to x_1, and an edge from x_i to x_{i+1} for $i \leqslant n+1$. We call x_0 the *top* of $[n]$.

Let $S \subseteq \omega$; the directed graph $[S]$ consists of one copy of $[s]$ for every $s \in S$, with all tops identified.

Definition 4.1.6. Two tops n_0 and n_1 of G_i^s are *connected* if there is an element $l \in G_i^s$ such that (n_0, l), (l, n_0), (n_1, l), (l, n_1) are edges in G_i^s. In this case l is called the *linking element*.

"To connect two tops n_0, n_1 of G_i^s using a linking element" means to add one new element l as well as the edges (n_0, l), (l, n_0), (n_1, l), (l, n_1) to the graph G_i^s.

A *component* is a maximal subgraph isomorphic to $[S]$ for some $S \subseteq \omega$. Note that this is not necessarily the same as a maximal connected component.

As an example, Figure 4.1 below shows the structure $[S]$ for $S = \{1, 3\}$, and Figure 4.2 shows two tops n_0 and n_1 connected via a linking element l.

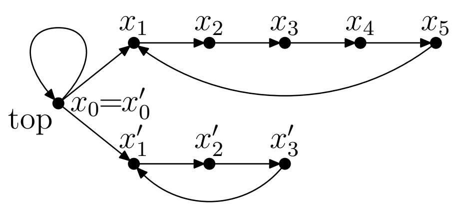

Figure 4.1: The structure $[\{1, 3\}]$.

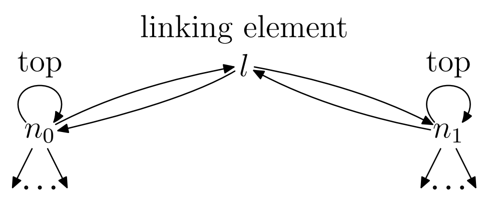

Figure 4.2: A linking element.

The construction of G_0 and G_1 is now as follows.

Step 0. Let $G_0^0 = G_1^0 = \{2n : n \in \omega\}$ and, for every $n \in \omega$, connect $2n$ to itself in both G_0^0 and G_1^0. Thus $2n$ is a top in G_0 and G_1.

Step s + 1. For $i \in \{0,1\}$, do the following. Let k be the unique element that enters some A_n^i at step s. Consider the component of G_i^s isomorphic to $[A_{n,s}^i]$ and containing the top $2n$. Expand this component to one isomorphic to $[A_{n,s}^i \cup \{k\}] = [A_{n,s+1}^i]$. If k is not the first element that enters A_n^i, then find the least m such that $2n$ is not connected to $2m$ in G_i^s and connect $2n$ to $2m$ using one new linking element.

Now, for every pair u_0, v_0 of tops in G_0^s and every pair u_1, v_1 of tops in G_1^s such that u_0, u_1 belong to the components isomorphic to $[S_0]$ and v_0, v_1 belong to the ones isomorphic to $[S_1]$ for some non-empty sets S_0 and S_1, do the following. Check if u_0, v_0 are connected in G_0^s, but u_1, v_1 are not connected in G_1^s, or vice versa. If *yes*, connect those tops u_i, v_i using one linking element which are not connected in G_i^s.
End of the construction.

Lemma 4.1.7. *G_0 and G_1 are isomorphic.*

Proof. According to the construction, each top $2n$ in G_0 (resp. G_1) belongs to the component isomorphic to $[A_n^0]$ (resp. $[A_n^1]$). Since $\{A_n^0\}_{n\in\omega}$ and $\{A_n^1\}_{n\in\omega}$ are one-to-one enumerations of the same family, G_0 and G_1 consist of the same collection of components.

To finish the proof, we need to show that for every pair n_0, n_1 of tops in G_0 and every pair m_0, m_1 of tops in G_1, if for $i \in \{0,1\}$, n_i and m_i belong to the isomorphic components, then n_0, n_1 are connected in G_0 iff m_0, m_1 are connected in G_1. Suppose that n_0, n_1, m_0, m_1 is a counterexample to the above statement such that, for instance, n_0, n_1 are connected in G_0 and m_0, m_1 are not connected in G_1.

Let n_i, m_i be the tops of the components isomorphic to $[S_i]$, where $i \in \{0,1\}$. By the construction, if n is the top of an infinite component of G_i, then n is connected to all other tops in $\mathcal{G}_i$. Therefore, $[S_0]$ and $[S_1]$ are finite. Hence there is a step s_0 such that both $G_0^{s_0}$ and $G_1^{s_0}$ contain the components isomorphic to $[S_0]$, $[S_1]$ with tops n_0, n_1 and m_0, m_1 respectively, and n_0, n_1 are connected in $G_0^{s_0}$. Now, if m_0 and m_1 have not yet been connected, then they will be connected at the next step. This contradiction proves the lemma.

□

Lemma 4.1.8. *G_0 and G_1 are not computably isomorphic.*

Proof. Let $f : G_0 \to G_1$ be a computable isomorphism. Note that f maps tops to tops, and the component containing the top $2n$ in G_0 is isomorphic to the one containing the top $f(2n)$ in G_1. Therefore, the enumerations $\{A_n^0\}_{n\in\omega}$ and $\{A_n^1\}_{n\in\omega}$

are reducible to one another via the computable functions $h_0(n) = f(2n)/2$ and $h_1(n) = f^{-1}(2n)/2$, which contradicts our choice of $\{A_n^0\}_{n\in\omega}$ and $\{A_n^1\}_{n\in\omega}$.

□

Lemma 4.1.9. *Let H be a computable graph isomorphic to G, then H is computably isomorphic either to G_0 or to G_1. Thus G has computable dimension two.*

Proof. Since H is computable, there is a computable list $t_0 < t_1 < t_2 < \dots$ of the tops in H, where $<$ is the natural order on ω. The structure H gives rise to a one-to-one computable enumeration $\{A_n\}_{n\in\omega}$ of $\mathcal{F}$ such that $k \in A_n$ iff there is a subgraph of H isomorphic to $[k]$ containing t_n as its top.

Since $\mathcal{F}$ has exactly two nonequivalent one-to-one computable enumerations, $\{A_n\}_{n\in\omega}$ is equivalent either to $\{A_n^0\}_{n\in\omega}$ or $\{A_n^1\}_{n\in\omega}$. Suppose that $\{A_n\}_{n\in\omega}$ is equivalent to $\{A_n^0\}_{n\in\omega}$. We now construct a computable isomorphism h from H to G_0.

By our assumption, there is a computable function f such that $A_n = A^0_{f(n)}$ for all n. Note that f is a permutation of ω because $\{A_n\}_{n\in\omega}$ and $\{A_n^0\}_{n\in\omega}$ are one-to-one. Take any $k \in H$; the value of $h(k)$ is defined according to the following three cases:

1) If $k = t_n$ for some n, then $h(k) = 2f(n)$.
2) If k is the linking element between t_n and t_m, then $h(k)$ is the linking element between $2f(n)$ and $2f(m)$ in G_0. Note that such a linking element exists since $H \cong G_0$.
3) If k is neither a top nor a linking element, then there are m and t_n such that k belongs to the subgraph of H isomorphic to $[m]$ with the top t_n. Let l be the length of the unique path from t_n to k without repetitions. Now, $h(k)$ is the unique element of G_0 belonging to the subgraph isomorphic to $[m]$ with the top $2f(n)$ such that the length of the path from $2f(n)$ to $h(k)$ without repetitions is equal to l.

By the construction, $h : H \to G_0$ is an isomorphism. It is computable since, for a given $k \in H$, one can effectively find out which one of the cases 1), 2) or 3) holds and then effectively find the value of $h(k)$.

□

To show that G is prime we will use the following model-theoretic fact.

Proposition 4.1.10. *Let $\mathcal{A}$ be a model such that for every $a \in \mathcal{A}$, there is a formula $\varphi_a(x)$ in the language of $\mathcal{A}$ with the property that*

$$\mathcal{A} \models \forall z\ (\varphi_a(z) \leftrightarrow a = z).$$

Then $\mathcal{A}$ is the prime model of its theory.

Lemma 4.1.11. *G is prime.*

Proof. Due to Proposition 4.1.10, to prove that G is prime it suffices to show that for every $a \in G$, there is a formula $\varphi_a(x)$ in the language of directed graphs such that $G \models \forall x\ (\varphi_a(x) \leftrightarrow a = x)$. Let $E(x, y)$ be the edge relation on G.

By the construction, the top of every infinite component is connected to all other tops. On the other hand, the top of every finite component is not connected to all other tops. To see this, let $2n_0$ be the top of a finite component $[A^0_{n_0}]$ in G_0, and let n_1 be such that $A^0_{n_0} = A^1_{n_1}$. Hence, $2n_1$ is the top of a finite component isomorphic to $[A^0_{n_0}]$ in G_1. Consider the step s_0 by which we have constructed the components $[A^0_{n_0}]$ and $[A^1_{n_1}]$ in $G^{s_0}_0$ and $G^{s_0}_1$ respectively. Since $\mathcal{F}$ contains infinitely many singletons, there are k_0 and k_1, such that $2k_0$ and $2n_0$ are not connected in $G^{s_0}_0$, $2k_1$ and $2n_1$ are not connected in $G^{s_0}_1$, and $A^0_{k_0}$, $A^1_{k_1}$ are equal one-element sets. Then $2k_0$, $2n_0$ are not connected in G_0 as well as $2k_1$, $2n_1$ are not connected in G_1 because we do not connect $2k_0$ with any top when the only element of $A^0_{k_0}$ is enumerated in it, and the same is true for $2k_1$ in G_1.

First, let us define $\varphi_a(x)$ when a is a top. If a is the top of a finite component $[S]$, then $\varphi_a(x)$ states that $E(x, x)$ and x belongs to a subgraph isomorphic to $[n(S)]$, where $n(S)$ is the marker for the finite set $S \in \mathcal{F}$. If a is the top of an infinite component $[S]$, then $\varphi_a(x)$ states that $E(x, x)\ \&\ \forall y\ (E(y, y) \rightarrow$ "x and y are connected via a linking element"), and x belongs to a subgraph isomorphic to $[n(S)]$, where $n(S)$ is the marker for the infinite set $S \in \mathcal{F}$.

If a is a linking element between two tops u and v, then

$$\varphi_a(x) = \exists y\ \exists z\ (E(x, y) \wedge E(y, x) \wedge E(x, z) \wedge E(z, x) \wedge \varphi_u(y) \wedge \varphi_v(z)).$$

If a is neither a top nor a linking element, then let k, l and u be such that a belongs to the subgraph of G isomorphic to $[k]$ with the top u, and l is the length of the unique path from u to a without repetitions. In this case $\varphi_a(x)$ states that

$$\exists z\ (\varphi_u(z)\ \&\ \text{"}x \text{ belongs to a subgraph isomorphic to } [k] \text{ with top } z\text{"}\ \&$$
$$\text{"there is a path of length } l \text{ without repetitions from } z \text{ to } x\text{"}).$$

□

Theorem 4.1.1 now follows from Lemmas 4.1.7, 4.1.8, 4.1.9, and 4.1.11.

4.2 Codings into another structures

Hirschfeldt, Khoussainov, Shore, and Slinko [32] developed the technique for coding directed graphs into structures like symmetric, irreflexive graphs, partial orders, lattices, rings, 2-step nilpotent groups, and so on. These codings are effective in the sense that they preserve various interesting computability-theoretic properties of the structures such as the computable dimension, the degree spectra of the structures, and the spectra of relations on computable structures.

Our goal in this section is to show that in some cases these codings also preserve the model-theoretic property of being the prime model. For instance, let G be a graph such that every element of it is defined by a first order formula. We will show that the codings of G into a partial order, a lattice, and an integral domain preserve this property. Hence these structures will be the prime models of their theories. However, in the case of integral domains we will need to add finitely many constants.

Let G be the directed graph constructed in the previous section. First, we show how to encode G into a prime symmetric, irreflexive graph H_G of computable dimension two. We then encode H_G into a prime partial order, a lattice, and an almost prime integral domain preserving its computable dimension.

4.2.1 Symmetric, irreflexive graphs

Let G be an infinite, computable graph, and E be its edge relation. Without loss of generality we will assume that $|G| = \omega$. A computably presentable symmetric, irreflexive graph $H_G = (|H_G|, F)$ is defined as follows.

1. $|H_G| = \{a, a', b, b', b''\} \cup \{c_i, d_i, e_i : i \in \omega\}$.
2. $F(x, y)$ holds only in the following cases.
   (a) $F(a, a')$, $F(a', a)$, $F(b, b')$, $F(b', b)$, $F(b', b'')$, $F(b'', b')$.
   (b) For all $i \in \omega$,

   $$F(a, c_i) \text{ and } F(c_i, a), \qquad F(d_i, e_i) \text{ and } F(e_i, d_i),$$
   $$F(c_i, d_i) \text{ and } F(d_i, c_i), \qquad F(b, e_i) \text{ and } F(e_i, b).$$

   (c) If $E(i, j)$ then $F(c_i, e_j)$ and $F(e_j, c_i)$.

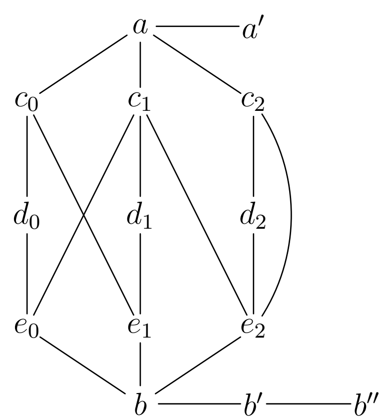

Figure 4.3: A portion of H_G.

Figure 4.3 shows a portion of the graph H_G in the case in which $E(0, 1)$, $E(1, 0)$, $E(1, 2)$, $E(2, 2)$, $\neg E(0, 0)$, $\neg E(0, 2)$, $\neg E(1, 1)$, $\neg E(2, 0)$, and $\neg E(2, 1)$.

Define

$$D(x) = \{x \in |H_G| : x \neq a' \wedge F(a,x)\} = \{c_i : i \in \omega\}$$

and

$$R(x,y) = \{(x,y) : D(x) \wedge D(y) \wedge \exists d, e(F(b,e) \wedge F(e,d) \wedge F(d,y) \wedge F(x,e))\}.$$

Then the mapping $g : i \to c_i$ is an isomorphism from G onto the graph with the domain D^{H_G} and the edge relation $R^{H_G}(x,y)$.

Proposition 4.2.1. *For any computable presentation of H_G, the sets $D^{H_G} = \{c_i^{H_G} : i \in \omega\}$, $\{d_i^{H_G} : i \in \omega\}$, $\{e_i^{H_G} : i \in \omega\}$, and the relation R^{H_G} are computable.*

Proof. Clearly, $D^{H_G} = \{c_i^{H_G} : i \in \omega\}$, $\{d_i^{H_G} : i \in \omega\}$, and $\{e_i^{H_G} : i \in \omega\}$ are computable since they are definable by quantifier-free formulas with parameters. Hence, R^{H_G} is also computable since for all $x, y \in D^{H_G}$,

$$\exists d, e[F(b,e) \wedge F(e,d) \wedge F(d,y) \wedge F(x,e)] \iff \forall d, e[(F(b,e) \wedge F(e,d) \wedge F(d,y)) \to F(x,e)].$$

□

Proposition 4.2.2. *The relations D and R are definable by first-order formulas in the language of graphs.*

Proof. It suffices to show that the constants a, a', b, b', and b'' are definable. Let

$$\psi_{b''}(x) = \exists!\, yF(x,y) \wedge \forall y[F(x,y) \to \exists!\, z(z \neq x \wedge F(z,y))],$$

then $\psi_{b''}$ defines b''. The following formulas define b', b, a', and a respectively:

$$\psi_{b'}(x) = \exists y(F(x,y) \wedge \psi_{b''}(y)), \qquad \psi_b(x) = \exists y(F(x,y) \wedge \psi_{b'}(y)) \wedge \neg\psi_{b''}(x),$$
$$\psi_{a'}(x) = \exists!\, yF(x,y) \wedge \neg\psi_{b''}(x), \qquad \psi_a(x) = \exists y(F(x,y) \wedge \psi_{a'}(y)).$$

□

Let G be the prime graph of computable dimension two constructed in Section 4.1, and let G_1, G_2 be its two computable presentations which are not computably isomorphic. For each $j = 1, 2$, let us choose a computable presentation H_{G_j} of H_G such that the isomorphic embedding $g_j : i \to c_i^{H_{G_j}}$ is computable.

Proposition 4.2.3. *H_G has computable dimension two.*

Proof. If $f : H_{G_1} \to H_{G_2}$ is a computable isomorphism, then so is $\widehat{f} = g_2^{-1} \circ f \circ g_1 : G_1 \to G_2$. Indeed, $E^{G_1}(i,j) \Leftrightarrow R^{H_{G_1}}(g_1(i), g_1(j)) \Leftrightarrow R^{H_{G_2}}(f \circ g_1(i), f \circ g_1(j)) \Leftrightarrow E^{G_2}(g_2^{-1} \circ f \circ g_1(i), g_2^{-1} \circ f \circ g_1(j))$. So, H_{G_1} and H_{G_2} are not computably isomorphic.

Let $H_{G'}$ be any computable presentation of H_G, and let G' be a computable graph with the domain $D^{H_{G'}}$ and the edge relation $R^{H_{G'}}$. Since $H_{G'} \cong H_G$ and D and R are definable relations, we have $G' \cong G$. Hence for some $j = 1, 2$, there is a computable isomorphism $h : G' \to G_j$. Now we can construct a computable isomorphism φ from $H_{G'}$ to H_{G_j}.

Let $\varphi(a^{H_{G'}}) = a^{H_{G_j}}$, $\varphi(a'^{H_{G'}}) = a'^{H_{G_j}}$, $\varphi(b^{H_{G'}}) = b^{H_{G_j}}$, $\varphi(b'^{H_{G'}}) = b'^{H_{G_j}}$, $\varphi(b''^{H_{G'}}) = b''^{H_{G_j}}$. For every other $x \in |H_{G'}|$, $\varphi(x)$ is defined as follows.

(1) If $x \in D^{H_{G'}}$, that is $x = c_i^{H_{G'}}$ for some $i \in \omega$, let $\varphi(x) = g_j(h(x)) = c_{h(x)}^{H_{G_j}}$.

(2) If $x = d_i^{H_{G'}}$ for some $i \in \omega$, let $\varphi(x) = d_{h(y)}^{H_{G_j}}$, where $y = c_i^{H_{G'}}$ is an element of $D^{H_{G'}}$ which is connected to x. In other words, $\varphi(x)$ is an element of $\{d_i^{H_{G_j}} : i \in \omega\}$ that is connected to $g_j(h(y)) = c_{h(y)}^{H_{G_j}}$.

(3) If $x = e_i^{H_{G'}}$ for some $i \in \omega$, let $\varphi(x) = e_{h(y)}^{H_{G_j}}$, where $y = c_i^{H_{G'}}$.

It is easy to check that this construction for $\varphi : H_{G'} \to H_{G_j}$ is effective. Therefore, H_G has dimension two.

□

Proposition 4.2.4. *H_G is prime.*

Proof. It suffices to show that every element $x \in |H_G|$ is definable by a first order formula. The formulas that define the constants a, a', b, b', and b'' are given in the proof of Proposition 4.2.2. Consider c_i; we know that there exists a formula $\varphi_i(x)$ that defines the element $i \in |G|$. Let $\psi_{c_i}(x)$ be the formula obtained from $\varphi_i(x)$ by replacing every occurrence of the binary predicate E with the formula for R, every occurrence of $\forall z \ldots$ with $\forall z(D(z) \to \ldots)$, and every occurrence of $\exists z \ldots$ with $\exists z(D(z) \wedge \ldots)$, where z is any variable. Then $\psi_{c_i}(x)$ defines c_i. Furthermore, d_i is defined by

$$\psi_{d_i}(x) = \neg\psi_a(x) \wedge \exists y(F(x,y) \wedge \psi_{c_i}(y)) \wedge \neg\exists y(F(x,y) \wedge \psi_b(y)),$$

and e_i is defined by $\psi_{e_i}(x) = \exists y(F(x,y) \wedge \psi_{d_i}(y)) \wedge \neg\psi_{c_i}(x)$. Therefore, H_G is prime.

□

4.2.2 Partial orderings

Let G be an infinite, computable, symmetric, irreflexive graph, and E be its edge relation. Again we assume that $|G| = \omega$. A computably presentable partial ordering $P_G = (|P_G|, \preccurlyeq)$ is defined as follows.

1. $|P_G| = \{a, b\} \cup \{c_i : i \in \omega\} \cup \{d_{i,j} : i < j \in \omega\}$.
2. The relation $\preccurlyeq$ is the smallest partial ordering on $|P_G|$ satisfying the following conditions.

(a) $a \preccurlyeq c_i \preccurlyeq b$ for all $i \in \omega$.

(b) If $i < j$ and $E(i,j)$, then $d_{i,j} \preccurlyeq c_i, c_j$.

(c) If $i < j$ and $\neg E(i,j)$, then $c_i, c_j \preccurlyeq d_{i,j}$.

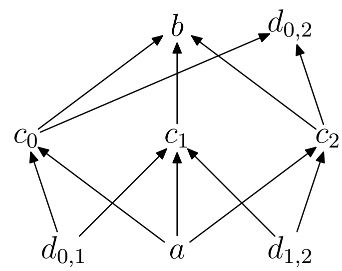

Figure 4.4: A portion of P_G.

Figure 4.4 shows a portion of the partial ordering P_G in the case in which $E(0,1)$, $E(1,2)$, and $\neg E(0,2)$.

Define

$$D(x) = \{x \in |P_G| : a \prec x \prec b\} = \{c_i : i \in \omega\}$$

and

$$R(x,y) = \{(x,y) : x \neq y \wedge D(x) \wedge D(y) \wedge \exists z \neq a\,(z \preccurlyeq x, y)\}.$$

Note that $g : i \to c_i$ is an isomorphism from G onto the graph with the domain D^{P_G} and the edge relation $R^{P_G}(x,y)$.

Proposition 4.2.5. *For any computable presentation of P_G, the relations D^{P_G} and R^{P_G} are computable.*

Proof. Obviously, D^{P_G} is computable, and so is R^{P_G} since for all $x \neq y \in D^{P_G}$,

$$\exists z \neq a\,(z \preccurlyeq x, y) \Longleftrightarrow \neg\exists z \neq b\,(x, y \preccurlyeq z).$$

□

Proposition 4.2.6. *The relations D and R are definable by first-order formulas in the language of partial orders.*

Proof. It suffices to show that the constants a and b are definable. Let $\psi_a(x) = \forall y\,(\exists z\,(z \prec y) \to x \preccurlyeq y)$ and $\psi_b(x) = \forall y\,(\exists z\,(y \prec z) \to y \preccurlyeq x)$. It is not hard to see that $\psi_a(x)$ and $\psi_b(x)$ define a and b, respectively.

□

Let G be the prime, symmetric, irreflexive graph of computable dimension two constructed in Section 4.2.1, and let G_1, G_2 be its two computable presentations which are not computably isomorphic. For each $j = 1, 2$, let us choose a computable presentation of P_{G_j} such that the mapping $g_j : i \to c_i^{P_{G_j}}$ is computable.

Proposition 4.2.7. *P_G has computable dimension two.*

Proof. If $f : P_{G_1} \to P_{G_2}$ is a computable isomorphism, then so is $\widehat{f} = g_2^{-1} \circ f \circ g_1 : G_1 \to G_2$. Therefore, P_{G_1} and P_{G_2} are not computably isomorphic.

Let $P_{G'}$ be any computable presentation of P_G, and let G' be a computable graph with the domain $D^{P_{G'}}$ and the edge relation $R^{P_{G'}}$. Since $P_{G'} \cong P_G$ and D and R are definable relations, we have $G' \cong G$. Hence for some $j = 1, 2$, there is a computable isomorphism $h : G' \to G_j$. A computable isomorphism φ from $P_{G'}$ to P_{G_j} is now defined as follows.

Let $\varphi(a^{P_{G'}}) = a^{P_{G_j}}$ and $\varphi(b^{P_{G'}}) = b^{P_{G_j}}$. For every $x \in D^{P_{G'}}$, that is if $x = c_i^{P_{G'}}$ for some $i \in \omega$, let $\varphi(x) = g_j(h(x)) = c_{h(x)}^{P_{G_j}}$. If $x = d_{i,j}^{P_{G'}}$ for some $i < j$, then either $\exists\, y_1, y_2 \in D^{P_{G'}}(y_1 \neq y_2 \wedge x \preccurlyeq y_1, y_2)$ or $\exists\, y_1, y_2 \in D^{P_{G'}}(y_1 \neq y_2 \wedge y_1, y_2 \preccurlyeq x)$. We can effectively find out which one of the cases holds as well as the relevant y_1, y_2. Suppose $x \preccurlyeq y_1, y_2$; in this case let $\varphi(x)$ be the unique element of P_{G_j} that is less than both $g_j(h(y_1)) = c_{h(y_1)}^{P_{G_j}}$ and $g_j(h(y_2)) = c_{h(y_2)}^{P_{G_j}}$ and that is not equal to $a^{P_{G_j}}$.

It is easy to see that this construction for $\varphi : P_{G'} \to P_{G_j}$ is effective. Therefore, P_G has computable dimension two.

□

Proposition 4.2.8. *P_G is prime.*

Proof. Let us show that every element of P_G is definable by a first order formula. The formulas that define the constants a and b are given in the proof of Proposition 4.2.6. Recall that every $i \in |G|$ is defined by some formula $\varphi_i(x)$. Now, every c_i is defined by the formula $\psi_{c_i}(x)$ obtained from $\varphi_i(x)$ by replacing every occurrence of the binary predicate E with the formula for R, every occurrence of $\forall z \ldots$ with $\forall z(D(z) \to \ldots)$, and every occurrence of $\exists z \ldots$ with $\exists z(D(z) \wedge \ldots)$, where z is any variable. If $E(i, j)$, then $d_{i,j}$ is defined by

$$\psi_{d_{i,j}}(x) = \neg\psi_a(x) \wedge \exists\, y_1, y_2(\psi_{c_i}(y_1) \wedge \psi_{c_j}(y_2) \wedge x \preccurlyeq y_1, y_2).$$

If $\neg E(i, j)$, then $d_{i,j}$ is defined by

$$\psi_{d_{i,j}}(x) = \neg\psi_b(x) \wedge \exists\, y_1, y_2(\psi_{c_i}(y_1) \wedge \psi_{c_j}(y_2) \wedge y_1, y_2 \preccurlyeq x).$$

Therefore, P_G is prime.

□

4.2.3 Lattices

Let G be an infinite, computable, symmetric, irreflexive graph with edge relation E and $|G| = \omega$. A computably presentable lattice $L_G = (|L_G|, \curlywedge, \curlyvee)$ is defined as follows.

1. $|L_G| = \{a, b, k\} \cup \{c_i, m_i : i \in \omega\} \cup \{d_{i,j} : i < j \wedge E(i, j)\}$.

2. For all $x, y \in |L_G|$, if $x \neq y$, then $x \curlyvee y = a$ and $x \curlywedge y = b$, except as required to satisfy the following conditions.

   (a) If $i < j$ and $E(i,j)$, then $c_i \curlyvee c_j = d_{i,j}$.

   (b) $k \curlyvee c_i = m_i$ for all $i \in \omega$.

   (c) $x \curlyvee b = x$ and $x \curlywedge a = x$ for all $x \in |L_G|$.

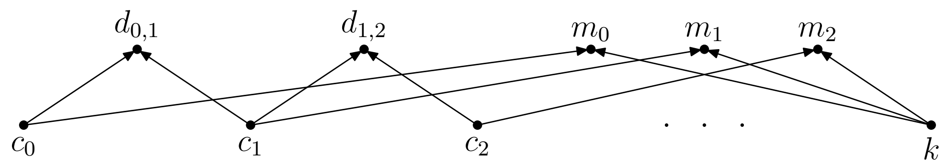

Figure 4.5: A portion of L_G.

Figure 4.5 shows a portion of the lattice L_G in the case in which $E(0,1)$, $E(1,2)$, and $\neg E(0,2)$. For simplicity, the top element a and the bottom element b of the lattice are not shown in the picture.

Define

$$D(x) = \{x \in |L_G| : (k \curlyvee x \neq a) \wedge (k \curlyvee x \neq x) \wedge x \neq b\} = \{c_i : i \in \omega\}$$

and

$$R(x,y) = \{(x,y) : x \neq y \wedge D(x) \wedge D(y) \wedge (x \curlyvee y \neq a)\}.$$

Note that $g : i \rightarrow c_i$ is an isomorphism from G onto the graph with the domain D^{L_G} and the edge relation $R^{L_G}(x,y)$. The following proposition is obvious.

Proposition 4.2.9. *For any computable presentation of L_G, the relations D^{L_G} and R^{L_G} are computable.*

Let G be the prime, symmetric, irreflexive graph of computable dimension two constructed in Section 4.2.1. If we add one isolated vertex to G, then the new graph will have the same computable dimension as G, and every element will be definable by a first order formula. This is because G does not have isolated vertices. So, in this section we assume that G has one isolated vertex. Now, let G_1, G_2 be two computable presentations of G that are not computably isomorphic. For each $j = 1, 2$, let us choose a computable presentation of L_{G_j} such that the mapping $g_j : i \rightarrow c_i^{L_{G_j}}$ is computable.

Proposition 4.2.10. *The relations D and R are definable by first-order formulas in the language of lattices.*

Proof. It suffices to show that the constants a, b, and k are definable. The formulas $\psi_a(x) = \forall y(x \curlyvee y = x)$ and $\psi_b(x) = \forall y(x \curlywedge y = x)$ define a and b, respectively. Since G has an isolated vertex, k is the only level-2 element of L_G whose join with any level-2 element is not a. The level-2 elements of L_G are $\{k, c_i : i \in \omega\}$. This can be expressed by the formula

$$\psi_k(x) = \exists z\, (\psi_a(z) \wedge lev_2(x) \wedge \forall y\, (lev_2(y) \to x \curlyvee y \neq z)),$$

where $lev_2(x) = \exists! y\, (x \neq y \wedge x \curlywedge y = y)$.

□

Proposition 4.2.11. *L_G has computable dimension two.*

Proof. If $f : L_{G_1} \to L_{G_2}$ is a computable isomorphism, then so is $\widehat{f} = g_2^{-1} \circ f \circ g_1 : G_1 \to G_2$. Therefore, L_{G_1} and L_{G_2} are not computably isomorphic.

Let $L_{G'}$ be any computable presentation of L_G, and let G' be a computable graph with the domain $D^{L_{G'}}$ and the edge relation $R^{L_{G'}}$. Since $L_{G'} \cong L_G$ and D and R are definable relations, we have $G' \cong G$. Hence for some $j = 1, 2$, there is a computable isomorphism $h : G' \to G_j$. A computable isomorphism φ from $L_{G'}$ to L_{G_j} is defined as follows.

Let $\varphi(a^{L_{G'}}) = a^{L_{G_j}}$, $\varphi(b^{L_{G'}}) = b^{L_{G_j}}$, and $\varphi(k^{L_{G'}}) = k^{L_{G_j}}$. For every other $x \in |L_{G'}|$, $\varphi(x)$ is defined as follows.

(1) If $x \in D^{L_{G'}}$, that is $x = c_i^{L_{G'}}$ for some $i \in \omega$, let $\varphi(x) = g_j(h(x)) = c_{h(x)}^{L_{G_j}}$.

(2) If $x \notin D^{L_{G'}}$ and $x \curlyvee k^{L_{G'}} \neq a^{L_{G'}}$, that is $x = m_i^{L_{G'}}$ for some $i \in \omega$, then there is $z \in D^{L_{G'}}$ such that $z \curlyvee k^{L_{G'}} = x$. In this case let $\varphi(x) = k^{L_{G_j}} \curlyvee \varphi(z)$.

(3) If $x \notin D^{L_{G'}}$ and $x \curlyvee k^{L_{G'}} = a^{L_{G'}}$, that is $x = d_{i,j}^{L_{G'}}$ for some $i < j$, then there are $z_1, z_2 \in D^{L_{G'}}$ such that $z_1 \curlyvee z_2 = x$. In this case let $\varphi(x) = \varphi(z_1) \curlyvee \varphi(z_2)$.

It is easy to see that this construction for $\varphi : L_{G'} \to L_{G_j}$ is effective. Hence L_G has dimension two.

□

Proposition 4.2.12. *L_G is prime.*

Proof. We show that every element of L_G is definable by a first order formula. The formulas that define the constants a, b, and k are given in the proof of Proposition 4.2.10. Let $i \in |G|$ be defined by a formula $\varphi_i(x)$, then c_i is defined by the formula $\psi_{c_i}(x)$ obtained from $\varphi_i(x)$ by replacing every occurrence of the binary predicate E with the formula for R, every occurrence of $\forall z \ldots$ with $\forall z(D(z) \to \ldots)$, and every occurrence of $\exists z \ldots$ with $\exists z(D(z) \wedge \ldots)$, where z is any variable. Each $d_{i,j}$ is defined by

$$\psi_{d_{i,j}}(x) = \exists\, z_1, z_2((x = z_1 \curlyvee z_2) \wedge \psi_{c_i}(z_1) \wedge \psi_{c_j}(z_2)).$$

Each m_i is defined by

$$\psi_{m_i}(x) = \exists\, z_1, z_2((x = z_1 \curlyvee z_2) \wedge \psi_{c_i}(z_1) \wedge \psi_k(z_2)).$$

Therefore, L_G is prime.

□

4.2.4 Integral domains

Definition 4.2.13. We say that a model $\mathcal{A}$ is *almost prime* if there is a finite tuple $\bar{a} = a_1, \ldots, a_k$ of elements of $\mathcal{A}$ such that the enriched structure $(\mathcal{A}, \bar{a})$ is the prime model of its own theory.

Let G be a computable symmetric, irreflexive graph with the edge relation E and $|G| = \omega$. Fix a number p which is either 0 or prime. We will use the convention that $\mathbb{Z}_0 = \mathbb{Z}$. Let I be the set of invertible elements of $\mathbb{Z}_p$, which is obviously finite.

The computably presentable integral domain A_G is defined to be

$$\mathbb{Z}_p[x_i : i \in \omega]\Big[\frac{y}{x_i x_j} : E(i,j)\Big]\Big[\frac{z}{x_i x_j} : \neg E(i,j)\Big]\Big[\frac{y}{x_i^n} : i, n \in \omega\Big].$$

From [32] it follows that A_G has the same computable dimension as G if G has the following property: for every finite set of nodes S, there exist nodes $x, y \notin S$ that are connected by an edge. Note that the graph constructed in Section 4.2.1 satisfies this property. Therefore, for this G, A_G has computable dimension two.

We prove that A_G is almost prime. This will require the following model-theoretic fact, which is a strengthening of Proposition 4.1.10.

Proposition 4.2.14. *Let $\mathcal{A}$ be a model in a countable language. Suppose that for every $a \in \mathcal{A}$, there is a formula $\varphi_a(x)$ in the language of $\mathcal{A}$ such that $\mathcal{A} \vDash \varphi_a(a)$ and $\varphi_a(\mathcal{A}) = \{b \in \mathcal{A} : \mathcal{A} \vDash \varphi_a(b)\}$ is finite. Then $\mathcal{A}$ is the prime model of its theory.*

Define

$$D(x) = \{x \in |A_G| : x \notin I \wedge \exists r(x^2 r = z)\},$$
$$Q(x, x') = \{(x, x') : D(x) \wedge \exists a \in I(x' = ax)\},$$

and

$$R(x, x') = \{(x, x') : D(x) \wedge D(x') \wedge \neg Q(x, x') \wedge \exists r(rxx' = y)\}.$$

Let $\varphi_i(x)$ be a formula that defines $i \in |G|$, and let $\psi_i(x)$ be the formula obtained from $\varphi_i(x)$ by replacing every occurrence of the binary predicate E with the formula for R, every occurrence of the equality relation with the formula for Q, and every occurrence of $\forall z \ldots$ and $\exists z \ldots$ with $\forall z(D(z) \to \ldots)$ and $\exists z(D(z) \wedge \ldots)$, respectively, where z is any variable.

From Lemmas 5.1, 5.2 and Corollary 5.5 of [32] it follows that I can be defined as the set of invertible elements of A_G, $D^{A_G} = \{ax_i : i \in \omega \wedge a \in I\}$, and $R^{A_G} = \{(ax_i, bx_j) : E^G(i,j) \wedge a, b \in I\}$. This means that Q^{A_G} is a congruence relation on (D^{A_G}, R^{A_G}), and the quotient structure of (D^{A_G}, R^{A_G}) modulo Q^{A_G} is isomorphic to (G, E). Therefore, $\psi_i(A_G) = \{ax_i : a \in I\}$. Note that $\psi_i(A_G)$ is finite since so is I.

Let

$$Gen = \{\pm 1\} \cup \{x_i : i \in \omega\} \cup \Big\{\frac{y}{x_i x_j} : E(i,j)\Big\} \cup \Big\{\frac{z}{x_i x_j} : \neg E(i,j)\Big\} \cup \Big\{\frac{y}{x_i^n} : i, n \in \omega\Big\}.$$

Every element of A_G can be expressed as a sum of products of elements of Gen. Let us add the constants for y and z to the language of rings. Now, for every $g \in Gen$, there is a formula $\psi_g(x)$ in the expanded language such that $A_G \vDash \psi_g(g)$ and $\psi_g(A_G)$ is finite. The formulas for x_i's are given above. For $y/x_i x_j$ the required formula is $\psi(x) = \exists u_1 \exists u_2 \ (\psi_i(u_1) \wedge \psi_j(u_2) \wedge u_1 u_2 x = y)$. It is easy to see that $\psi(A_G)$ is finite. The other cases are similar.

Since every $a \in A_G$ can be expressed as a term involving elements of Gen, one can construct a formula $\psi_a(x)$ in the language expanded by new constants for y and z such that $\psi_a(x)$ holds on a in A_G and $\psi_a(A_G)$ is finite. Therefore, due to Proposition 4.2.14, A_G is almost prime.

Chapter 5
Π^0_1-presentations of algebras

In this chapter we study the question as to which computable algebras possess non-computable Π^0_1-presentations. The background for this subject is provided in the General Introduction.

Here is a brief outline of the chapter. In the first section, we give the basic definitions of computable, Σ^0_1-, and Π^0_1-algebras, and provide some examples. In the second section we provide a theorem that characterizes those Σ^0_1-algebras that are non-computable. As a corollary we obtain that the isomorphism types of finitely generated computable algebras (such as arithmetic or term algebras) and of infinite computable fields fail to have non-computable Σ^0_1-presentations. In the third section we single out a class of algebras and call the algebras from that class term-separable. We prove that many well-known algebras such as arithmetic, term algebras, infinite fields and vector spaces are term-separable. Finally, the last section is devoted to showing that all computable term-separable algebras are isomorphic to non-computable Π^0_1-algebras.

5.1 Preliminaries

We now turn to the basic notions of this chapter. For the basics of computability theory the reader is referred to Soare [71]. An *algebra* is a structure of a finite purely functional language (signature) σ. Thus, any algebra $\mathcal{A}$ is of the form $(A; f_0^{\mathcal{A}}, \ldots, f_r^{\mathcal{A}})$, where A is a nonempty set called the domain of the algebra, and each $f_i^{\mathcal{A}}$ is a total operation on the domain A that interprets the function symbol $f_i \in \sigma$. When there is no confusion the operation named by f_i is denoted by the same symbol f_i. We refer to the symbols $f_0, \ldots, f_r$ as the signature of the algebra. Often we call the operations $f_0, \ldots, f_r$ *basic operations* or *functions* (of the algebra $\mathcal{A}$). Presburger arithmetic $(\omega; 0, S, +)$ is an algebra, so are groups, rings, lattices, and Boolean algebras. Fundamental structures which arise in computer science such as lists, stacks, queues, trees, and vectors can all be viewed and studied as algebras.

We now define the notion of a term of an algebra $\mathcal{A}$ over a variable set $X = \{x_0, x_1, \ldots\}$.

Definition 5.1.1. Let $\mathcal{A} = (A; f_0, \ldots, f_r)$ be an algebra. We define the terms of this algebra as formal expressions over a variable set X and the domain A as follows. Every element $a \in A$ and variable $x \in X$ is a term. If $t_1, \ldots, t_n$ are terms and $f \in \sigma$ is a function symbol of arity n, then $f(t_1, \ldots, t_n)$ is also a term.

As the terms are formal expressions formed from the set $A \cup X$ using the signature σ, it makes sense to talk about *syntactic equality* between terms of the algebra $\mathcal{A}$. For instance, examples of terms of arithmetic $(\omega, S, +, \times)$ are 5, $(x +$

$(7 \times y)) + S(6)$, $2+7$ and $7+2$. Note that syntactically the terms $2+7$ and $7+2$ are distinct. The elements $\bar{a}$ appearing in a term t of the algebra $\mathcal{A}$ are called parameters of t. We write $t(\bar{x}, \bar{a})$ to mean that the variables of the term t are among $\bar{x}$ and the parameters are among $\bar{a}$.

Consider the set of all terms without parameters. It can be transformed into an algebra in a natural way by declaring the value of f on any tuple $(t_1, \dots, t_n)$ to be the term $f(t_1, \dots, t_n)$. This is called the *term algebra* with generator set X.

Let $\mathcal{A} = (A, f_0, \dots, f_n)$ be an algebra with a computable universe. For each term $t = t(\bar{a})$ of $\mathcal{A}$ without free variables, introduce a new constant c_t that names the element $t(\bar{a})$. Expand the signature σ by adding to it all these constant symbols. So, elements $a \in A$ may have several constants c naming it. Denote the expanded signature by σ_A. Thus, we have an expansion of $\mathcal{A}$ by the constants c_t.

Definition 5.1.2. Consider the expanded algebra $\mathcal{A}$ in the signature σ_A.

1. The *atomic diagram of* $\mathcal{A}$, denoted by $D(\mathcal{A})$, is the set of all expressions of the type $f_i(c_{a_1}, \dots, c_{a_n}) = f_j(c_{b_1}, \dots, c_{b_k})$, $f_i(c_{a_1}, \dots, c_{a_n}) = c_b$, $c_a = c_b$ and their negations which are true in the algebra $\mathcal{A}$. The algebra $\mathcal{A} = (A; f_0, \dots, f_n)$ is *computable* if its atomic diagram is a computable set.
2. The *positive atomic diagram of* $\mathcal{A}$, denoted by $PD(\mathcal{A})$, is the set of all expressions of the type $f_i(c_{a_1}, \dots, c_{a_n}) = f_j(c_{b_1}, \dots, c_{b_k})$, $f_i(c_{a_1}, \dots, c_{a_n}) = c_b$, and $c_a = c_b$ which are true in the algebra $\mathcal{A}$. The algebra $\mathcal{A} = (A; f_0, \dots, f_n)$ is Σ^0_1-*algebra* if its positive atomic diagram is a computably enumerable set.
3. The *negative atomic diagram of* $\mathcal{A}$, denoted by $ND(\mathcal{A})$, is the set of all expressions of the type $f_i(c_{a_1}, \dots, c_{a_n}) \neq f_j(c_{b_1}, \dots, c_{b_k})$, $f_i(c_{a_1}, \dots, c_{a_n}) \neq c_b$, and $c_a \neq c_b$ which are true in the algebra $\mathcal{A}$. The algebra $\mathcal{A} = (A; f_0, \dots, f_n)$ is Π^0_1-*algebra* if its negative atomic diagram is a computably enumerable set.

It is easy to see that the algebra is computable if and only if it is both Σ^0_1- and Π^0_1-algebra. We give now several examples.

Example 5.1.3. Let $\mathcal{A} = (A; f_0, \dots, f_r)$ be an infinite computable algebra. Then it is isomorphic to an algebra $(\omega, h_1, \dots, h_r)$, where each h_i is a computable function. Clearly all algebras of the form $(\omega, g_1, \dots, g_r)$, where each g_i is a computable function, are computable.

Example 5.1.4. Typical examples of Σ^0_1-algebras are:

(i) The Lindenbaum algebras of computably enumerable first-order theories, such as Peano arithmetic.

(ii) All finitely presented groups and, in fact, all finitely presented algebras.

The following two examples provide simple ways of building Π^0_1-algebras.

Example 5.1.5. Let $p_1, \ldots, p_n$ be computable permutations of ω. Consider the group G generated by these permutations. Then G is a Π^0_1-algebra. Indeed, if g and g' are elements of this group, then their non-equality is confirmed by the existence of an $n \in \omega$ at which $g(n) \neq g'(n)$.

Example 5.1.6. Let $\mathcal{A} = (\omega, f_0, \ldots, f_r)$ be a computable algebra. For the terms $t(\bar{x})$ and $p(\bar{x})$, we write $t =_{\mathcal{A}} p$ if the values of t and p are equal for all instantiations of the variables. Consider the algebra $\mathcal{B}$ obtained by factoring the term algebra with respect to the relation $=_{\mathcal{A}}$. The algebra $\mathcal{B}$ is a Π^0_1-algebra since non-equality between any two terms $t(\bar{x})$ and $p(\bar{x})$ is confirmed by the existence of a tuple $\bar{a} \in A$ at which $t(\bar{a}) \neq p(\bar{a})$.

Example 5.1.7. Let $\Sigma = \{0, \ldots, k-1\}$ be a finite alphabet and $L \subseteq \Sigma^*$ be a computable language. Consider a computable algebra $\mathcal{A} = (\Sigma^*, S_0, \ldots, S_{k-1})$, where $S_i(x) = xi$ for every x. Define a congruence relation $\sim_L$ on $\mathcal{A}$ as follows: $x \sim_L y$ iff $\forall u\ (xu \in L \iff yu \in L)$. Then $\mathcal{A}/\sim_L$ is a Π^0_1-algebra.

A Π^0_1-algebra (or Σ^0_1-algebra) $\mathcal{A}$ can be explained as follows. As the negative atomic diagram (or the positive atomic diagram, respectively) of $\mathcal{A}$ can be computably enumerated, the set $E = \{(c_a, c_b) \mid c_a = c_b \text{ is true in the algebra } \mathcal{A}\}$, representing the equality relation in $\mathcal{A}$, is the complement of a c.e. set (or is a c.e. set, respectively). Let f be a basic n-ary operation on $\mathcal{A}$. From the definition of a computably enumerable algebra, the operation f can be thought of as a function induced by a computable function, often also denoted by f, which *respects* the E-equivalence classes in the following sense: for all $x_1, \ldots, x_n, y_1, \ldots, y_n$ if $(x_i, y_i) \in E$, then $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in E$. Therefore, a natural way to think about $\mathcal{A}$ is that the elements of $\mathcal{A}$ are E-equivalence classes, and the operations of $\mathcal{A}$ are induced by computable operations. This reasoning suggests another equivalent approach to the definition of a Π^0_1-algebra (as well as a Σ^0_1-algebra) explained in the next paragraph.

Let E be an equivalence relation on ω. A computable n-ary function f *respects* E if for all natural numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ such that $(x_i, y_i) \in E$ for $i = 1, \ldots, n$, we have $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in E$. Let $\omega(E)$ be the factor set obtained by factorizing ω by E, and let $f_0, \ldots, f_r$ be computable operations on ω which respect the equivalence relation E. An *E-algebra* is then the algebra $(\omega(E), F_0, \ldots, F_r)$, where each F_i is naturally induced by f_i. It is now not hard to show that an algebra $\mathcal{A}$ is a Π^0_1-algebra if and only if $\mathcal{A}$ is an E-algebra for some Π^0_1 equivalence relation E. In a similar way, an algebra $\mathcal{A}$ is a Σ_1-algebra if and only if $\mathcal{A}$ is an E-algebra for some computably enumerable equivalence relation E.

The *isomorphism type* of an algebra $\mathcal{A}$ is the set of all algebras isomorphic to $\mathcal{A}$. We are interested in those algebras whose isomorphism types contain Π^0_1-algebras. We formalize this in the following definitions. An algebra is Π^0_1-*presentable* if it is isomorphic to a Π^0_1-algebra. Note that there is a distinction between Π^0_1-algebras and Π^0_1-presentable algebras. Π^0_1-algebras are given explicitly by Turing machines representing the basic operations and the complement of the equality relation of the algebra, while Π^0_1-presentability refers to the property of the isomorphism types of algebras. All these notions make sense for Σ^0_1-presentable algebras as well, and we will use them without explicit definitions.

There are some notational conventions we need to make. Let $\mathcal{A}$ be a Π^0_1-algebra. As the equality relation on $\mathcal{A}$ can be thought of as an equivalence relation (with a c.e. complement) on ω, we can refer to the elements of $\mathcal{A}$ as natural numbers keeping in mind that each number n represents an equivalence class (that is, an element of $\mathcal{A}$). Thus, n can be regarded either as an element of $\mathcal{A}$, representing the equivalence class containing n, or the natural number n. The meaning which we use will be clear from the content. Sometimes we denote elements of $\mathcal{A}$ by $[n]$, with $[n]$ representing the equivalence class containing the number n.

5.2 Failing non-computable Σ^0_1-presentations

This section is for completeness and the main theorem is from [44]. However, we provide more applications of the theorem in order to contrast Σ^0_1- and Π^0_1-presentations of algebras in the last section.

Let $\mathcal{A}$ and $\mathcal{B}$ be Σ^0_1-algebras. A homomorphism h from the algebra $\mathcal{A}$ into the algebra $\mathcal{B}$ is called a *computable homomorphism* if there exists a computable function $f : \omega \to \omega$ such that h is induced by f. In other words, for all $n \in \omega$, we have $h([n]) = [f(n)]$. We call f a representation of h. Clearly, if h is a computable homomorphism, then its *kernel*, that is, the set $\{(n, m) \mid h([n]) = h([m])\}$, is computably enumerable. We say that h is *proper* if there are distinct $[n]$ and $[m]$ in $\mathcal{A}$ whose images under h coincide. In this case the image $h(\mathcal{A})$ is called a proper homomorphic image of $\mathcal{A}$.

Our goal is to give a syntactic characterization of Σ^0_1-algebras that are computable. Let $\mathcal{A}$ be a Σ^0_1-algebra. A *fact* is a computably enumerable conjunction $\&_{i\in\omega}\varphi_i(\bar{c}_i)$ of sentences, where each $\varphi_i(\bar{c}_i)$ is of the form $\forall\bar{x}\psi_i(\bar{x}, \bar{c}_i)$ with $\psi_i(\bar{x}, \bar{c}_i)$ being the negation of an atomic formula. Call all non-computable Σ^0_1-algebras *properly* Σ^0_1. For example, any finitely generated algebra with undecidable equality problem is properly Σ^0_1.

Definition 5.2.1. An algebra $\mathcal{A}$ *preserves the fact* $\&_{i\in\omega}\varphi_i(\bar{c}_i)$ if $\mathcal{A}$ satisfies the fact and there is a proper homomorphic image of $\mathcal{A}$ in which the fact is true.

The theorem below tells us that properly Σ^0_1-algebras possess many homomorphisms which are well behaved with respect to the facts true in $\mathcal{A}$.

Theorem 5.2.2. *A Σ^0_1-algebra $\mathcal{A}$ is properly Σ^0_1 if and only if $\mathcal{A}$ preserves all the facts true in $\mathcal{A}$.*

Proof. Assume that $\mathcal{A}$ is a computable algebra. We can make the domain of $\mathcal{A}$ to be ω. Thus, in the algebra $\mathcal{A}$, the fact $\&_{i\neq j}(i \neq j)$ is clearly true. This fact cannot be preserved in any proper homomorphic image of $\mathcal{A}$.

For the other direction, we first note the following. Given elements m and n of the algebra, it is possible to effectively enumerate the minimal congruence relation, denoted by $\eta(m,n)$, of the algebra which contains the pair (m,n). Now note that if $[m]=[n]$, then $\eta(m,n)$ is the equality relation in $\mathcal{A}$. Denote by $\mathcal{A}(m,n)$ the quotient algebra obtained by factorizing $\mathcal{A}$ by $\eta(m,n)$. Clearly, $\mathcal{A}(m,n)$ is computably enumerable.

Now assume that $\mathcal{A}$ is a properly Σ_1^0-algebra and $\&_{i\in\omega}\varphi_i(\bar{c}_i)$ is a fact true in $\mathcal{A}$ which cannot be preserved. Hence, for any m and n in the algebra, if $[m]\neq[n]$, then in the quotient algebra $\mathcal{A}(m,n)$ the fact $\&_{i\in\omega}\varphi_i(\bar{c}_i)$ cannot be satisfied. Therefore, for given m and n, there exists an i such that in the quotient algebra $\mathcal{A}(m,n)$ the sentence $\neg\varphi_i(\bar{c}_i)$ is true. Now the sentence $\neg\varphi_i(\bar{c}_i)$ is equivalent to an existential sentence quantified over a positive atomic formula. Note that the existential sentences quantified over positive atomic formulas true in $\mathcal{A}(m,n)$ can be computably enumerated. Hence, in the original algebra $\mathcal{A}$, for all m and n, either $[m]=[n]$ or there exists an i such that $\neg\varphi_i(\bar{c}_i)$ is true in $\mathcal{A}(m,n)$. This shows that the equality relation in $\mathcal{A}$ is computable, contradicting the assumption that $\mathcal{A}$ is a properly Σ_1^0-algebra. The theorem is proved.

□

There are several interesting corollaries of the theorem above.

Corollary 5.2.3. *If $\mathcal{A}$ is properly computably enumerable, then any two distinct elements m and n in $\mathcal{A}$ can be homomorphically mapped into distinct elements in a proper homomorphic image of $\mathcal{A}$.*

Proof. Indeed, take the fact $m\neq n$ true in $\mathcal{A}$ and apply the theorem.

□

Call two homomorphic images $h_1(\mathcal{A})$ and $h_2(\mathcal{A})$ of an algebra $\mathcal{A}$ *distinct* if the congruences induced by h_1 and h_2 are different.

Corollary 5.2.4. *If $\mathcal{A}$ is properly computably enumerable, then any fact true in $\mathcal{A}$ is true in infinitely many distinct homomorphic images of $\mathcal{A}$. In particular, $\mathcal{A}$ cannot have finitely many congruences.*

Proof. Let φ be a fact true in $\mathcal{A}$. By the theorem above, there is a homomorphic image $h_1(\mathcal{A})$ in which φ is true and distinct elements m_1 and n_1 in $\mathcal{A}$ for which $h_1(m_1)=h_1(n_1)$. Now consider the fact $\varphi\&(m_1\neq n_1)$ and apply the theorem to this fact. There is a homomorphic image $h_2(\mathcal{A})$ in which $\varphi\&(m_1\neq n_1)$ is true and distinct elements m_2 and n_2 in $\mathcal{A}$ for which $h_2(m_2)=h(n_2)$. Now consider the fact $\varphi\&(m_1\neq n_1)\&(m_2\neq n_2)$ and apply the theorem to this fact. The corollary now follows by induction.

□

This theorem can now be applied to provide several algebraic conditions for computable algebras not to have properly Σ_1^0-presentations.

Corollary 5.2.5. *In each of the following cases an infinite computably enumerable algebra $\mathcal{A}$ is computable:*

1) *There exists a c.e. sequence (x_i, y_i) such that $[x_i] \neq [y_i]$ for all i, and for any non-trivial congruence relation η, there is (x_j, y_j) for which $([x_j], [y_j]) \in \eta$.*
2) *$\mathcal{A}$ has finitely many congruences.*
3) *$\mathcal{A}$ is finitely generated and every non-trivial congruence relation on $\mathcal{A}$ has a finite index.*
4) *No computable field has a properly Σ_1^0-presentation.*
5) *No finitely generated computable algebra has a properly Σ_1^0-presentation.*

Proof. For part 1), we see that the fact $\&_{i\in\omega}[x_i] \neq [y_i]$ is true in $\mathcal{A}$. The assumption states that this fact cannot be preserved in any proper homomorphic image of $\mathcal{A}$. Hence $\mathcal{A}$ must be a computable algebra by the theorem above. For part 2), let $\eta_0,\ldots,\eta_k$ be all the non-trivial congruences of $\mathcal{A}$; for each η_i, take (x_i, y_i) such that $[x_i] \neq [y_i]$ and $([x_i], [y_i]) \in \eta_i$. Then the fact $\&_{i\leqslant k}([x_i] \neq [y_i])$ is true in $\mathcal{A}$ but cannot be preserved in any proper homomorphic image of $\mathcal{A}$. Thus, $\mathcal{A}$ is a computable algebra. For part 3), consider any two elements $[m]$ and $[n]$ in $\mathcal{A}$ and consider the congruence relation $\eta([m], [n])$ defined in the proof of the theorem. By the assumption, $[m] \neq [n]$ iff the algebra $\mathcal{A}(m, n)$ is finite. The set $X = \{(m, n) \mid \mathcal{A}(m, n) \text{ is finite}\}$ is computably enumerable. Hence, the fact $\&_{(m,n)\in X}([m] \neq [n])$ is true in $\mathcal{A}$ but cannot be preserved in any proper homomorphic image of $\mathcal{A}$. For part 4), consider a computable field $\mathcal{F} = (F; +, \times, 0, 1)$. This algebra has only two congruence relations (both are trivial). Hence by part 2), $\mathcal{F}$ does not have a proper Σ_1^0-presentation. For the last part, assume that $\mathcal{A}$ is a computable finitely generated algebra. Let $a_1, \ldots, a_n$ be the generators. Note that for any element $b \in A$, there exists a term t_b over the generating set $\{a_1, \ldots, a_n\}$ whose value in $\mathcal{A}$ equals b. Consider the following fact $\&_{b\neq c} t_b \neq t_c$. Clearly, this fact is true in the algebra but cannot be preserved in any proper homomorphic image of $\mathcal{A}$. Hence all Σ_1^0-presentations of $\mathcal{A}$ fail to be non-computable.

□

Note that from the corollary above, all finitely generated term algebras, infinite computable fields, and arithmetic fail to possesses non-computable Σ_1^0-presentations. The last section shows that all these algebras possess non-computable Π_1^0-presentations.

5.3 Term-separable algebras

In this section we define the notion of term-separable algebras and provide several examples of such algebras.

Definition 5.3.1. Let $\mathcal{A} = (A, f_1, \ldots, f_r)$ be an algebra. We say that $\mathcal{A}$ is *term-separable* if for every finite set of terms $\{t_1(x, y), \ldots, t_n(x, y)\}$ with parameters from A, every $J \subseteq \{1, \ldots, n\}^2$, and every $a \in A$, the following holds:

$$\mathcal{A} \vDash \bigwedge_{\langle k,l \rangle \in J} t_k(a, a) \neq t_l(a, a) \longrightarrow \exists b_1 \, \exists b_2 \, (b_1 \neq b_2) \wedge \bigwedge_{\langle k,l \rangle \in J} t_k(b_1, b_2) \neq t_l(b_1, b_2).$$

Proposition 5.3.2. *Let $\mathcal{A}$ be an infinite algebra and for every two terms $t_1(x)$ and $t_2(x)$ with parameters from A, the set $\{a \in A : \mathcal{A} \vDash t_1(a) = t_2(a)\}$ is either finite or equals A. Then $\mathcal{A}$ is term-separable.*

Proof. Consider a set of terms $t_1(x, y), \ldots, t_n(x, y)$ with parameters from A and a set $J \subseteq \{0, \ldots, n\}^2$ such that

$$\mathcal{A} \vDash \bigwedge_{\langle k,l \rangle \in J} t_k(a, a) \neq t_l(a, a).$$

Consider the terms $t_1(x, a), \ldots, t_n(x, a)$. For each $\langle k, l \rangle \in J$, let $B_{k,l} = \{b \in A : \mathcal{A} \vDash t_k(b, a) = t_l(b, a))\}$. Since $a \notin B_{k,l}$, $B_{k,l}$ is finite. Then there exists $b \in A \setminus \bigcup_{\langle k,l \rangle \in J} B_{k,l}$ such that $b \neq a$. Hence,

$$\mathcal{A} \vDash \bigwedge_{\langle k,l \rangle \in J} t_k(b, a) \neq t_l(b, a).$$

□

In the next proposition we provide several examples of term-separable algebras.

Proposition 5.3.3. *The following algebras are term-separable:*

1) arithmetic $(\omega, S, +, \times)$,

2) term algebras,

3) infinite fields,

4) torsion-free abelian group,

5) infinite vector spaces over a finite field.

Proof. For arithmetic or an infinite field, every term $t(x)$ with parameters is equivalent to a polynomial with coefficients from the set of natural numbers or from the field, respectively. Every non-zero polynomial has only finitely many zeros. Hence, the condition of Proposition 5.3.2 holds, and these algebras are term-separable. For part 2), consider two terms $t_1(x)$ and $t_2(x)$ such that $\mathcal{A} \vDash t_1(a) \neq t_2(a)$ for some

$a \in A$. Therefore, the terms $t_1(a)$ and $t_2(a)$ differ syntactically and, hence, $t_1(x)$ and $t_2(x)$ differ syntactically. So, $\mathcal{A} \models \forall b\ (t_1(b) \neq t_2(b))$ which implies that any term algebra is term-separable. For part 4), any term $t(x)$ is equal to the expression $nx + a$, where $n \in \mathbb{Z}$ and $a \in A$. Since the group is torsion-free, the equation $t(x) = 0$ has at most one solution if $n \neq 0$ or $a \neq 0$. The proof for the case of infinite vector spaces is similar to the above.

□

5.4 Admitting non-computable Π^0_1-presentations

This section is devoted to the proof of the following result.

Theorem 5.4.1. *Let $\mathcal{A} = (A; f_1, \ldots, f_r)$ be a computable term-separable algebra and $\mathbf{d}$ be any c.e. Turing degree. Then $\mathcal{A}$ possesses a Π^0_1-presentation of degree $\mathbf{d}$. In particular, it possesses a non-computable Π^0_1-presentation.*

Proof. We will construct the required Π^0_1-presentation of $\mathcal{A}$ step-by-step. At the end of step s, we have a number n_s and a collection of finite sets $\{C^s_i\}_{i\in\omega}$ given by their canonical indices such that $C^s_i \neq \varnothing$ for $i \leqslant n_s$, and $C^s_i = \varnothing$ for $i > n_s$. Also we have partial functions $h_1, \ldots, h_r$ with $\mathrm{dom}(h_i) \subseteq (\cup_{i\in\omega} C^s_i)^{m_i}$ and $\mathrm{range}(h_i) \subseteq \cup_{i\in\omega} C^s_i$, where m_i is the arity of f_i. Each h_i has the following property: if $\langle c^1_j, c^2_j\rangle \in \eta_s$ for all $j \leqslant m_i$, then $\langle h_i(\bar{c}^1), h_i(\bar{c}^2)\rangle \in \eta_s$, where

$$\forall\, x, y \in \cup_{i\in\omega} C^s_i \quad \langle x, y\rangle \in \eta_s \Longleftrightarrow \exists\, i\ \{x, y\} \subseteq C^s_i.$$

Furthermore, if $t_1(\bar{c}_1)$ and $t_2(\bar{c}_2)$ are terms constructed from the functions $h_1, \ldots,$ h_r with $\bar{c}_1, \bar{c}_2 \in \cup_{i\in\omega} C^s_i$ that differ syntactically, then their values are also different, provided that they are both defined.

Call $g \in \cup_{i\in\omega} C^s_i$ a *ground* element if for every term $t(\bar{x})$ constructed from the functions $h_1, \ldots, h_r$ such that $t(\bar{x})$ is not equal to some variable x or constant c, we have that $g \neq t(\bar{c})$ for every tuple $\bar{c} \in \cup_{i\in\omega} C^s_i$. Note that for every $d \in \cup_{i\in\omega} C^s_i$, there exists a unique term $t(\bar{c})$ constructed from the functions $h_1, \ldots, h_r$ with a tuple $\bar{c}$ of ground elements such that $d = t(\bar{c})$. We denote this term by $\widetilde{d}$. Note that if g is a ground element, then $\widetilde{g} = g$.

For each $i \leqslant n_s$, we have a triple of ground elements a_i, b_i, e_i that are all distinct. Initially $\{a_i, b_i, e_i\} \subseteq C^s_i$, but in some subsequent step a_i and b_i may move to other sets C^s_j, C^s_k, while e_i is always in C^s_i to ensure that this set will never be empty.

Also, the mapping $\psi_s : i \to C^s_i$ gives us a partial isomorphism between $\mathcal{A} \cap \{0, \ldots, n_s\}$ and $\{C^s_i\}_{i\leqslant n_s}$ in the following sense: for all $i \leqslant r$, for every tuple $a_1, \ldots, a_{m_i} \in \{0, \ldots, n_s\}$, and for every tuple $c_1, \ldots, c_{m_i}$ such that $c_j \in C^s_{a_j}$, if $h_i(\bar{c})$ is defined, then $f_i(\bar{a}) \leqslant n_s$ and $h_i(\bar{c}) \in C^s_{f_i(\bar{a})}$.

Define a function $g_s : \cup_{i\in\omega} C^s_i \to \omega$ such that $g_s(a) = i$ if $a \in C^s_i$. Let D be a c.e. set of degree $\mathbf{d}$ and let D^s denote the elements enumerated in D by the step

s. When we add a new element during the construction, we always take the least number that has not been used so far.

Step 0. Let $C_0^0 = \{a_0, b_0, e_0\}$ and $n_0 = 0$.

Step s + 1. This step has three substeps. At the end of substep l ($l = 1, 2, 3$), we will have constructed the sets $C_i^{s,l}$.

Case A. If for all $i \leqslant n_s$, $i \notin D^s$ or $g_s(a_i) \neq g_s(b_i)$, then

1) Let $n_{s+1} = n_s + 1$ and $C_i^{s,1} = C_i^s$ for $i \leqslant n_{s+1}$.
2) Put new (ground) elements $a_{n_{s+1}}, b_{n_{s+1}}, e_{n_{s+1}}$ into $C_{n_{s+1}}^{s,2}$ and let $C_i^{s,2} = C_i^{s,1}$ for $i \leqslant n_s$.
3) For every $i \leqslant r$, every tuple $a_1, \ldots, a_{m_i} \in \{0, \ldots, n_{s+1}\}$ such that $f_i(\bar{a}) \leqslant n_{s+1}$, and every tuple $c_1, \ldots, c_{m_i}$ such that $c_j \in C_{a_j}^{s,2}$, if $h_i(\bar{c})$ has not yet been defined, then add a new element to $C_{f_i(\bar{a})}^{s,3}$ and declare it to be the value of $h_i(\bar{c})$.

Let $C_i^{s+1} = C_i^{s,3}$ for all $i \leqslant n_{s+1}$.

Case B. If the condition of case A does not hold, then take the least i with the property $i \in D^s$ and $g_s(a_i) = g_s(b_i) = i$. Consider the set

$$D = \{t(\bar{c}) : \exists\, d \in \cup_{i\in\omega} C_i^s \text{ such that } \widetilde{d} = t(\bar{c})\}.$$

If $t(\bar{c}) \in D$, then let $t^*(x, y)$ be the term obtained from $t(\bar{c})$ by replacing each occurrence of a_i with x, each occurrence of b_i with y, every parameter c with $g_s(c)$, and every functional symbol h_i with f_i. For example, the terms $t_1 = a_i$ and $t_2 = b_i$ are in D. Then $t_1^* = x$ and $t_2^* = y$.

Let $D = \{t_1(\bar{c}_1), \ldots, t_n(\bar{c}_n)\}$ and $J = \{\langle k, l\rangle : \mathcal{A} \models\ t_k^*(i, i)\ \neq\ t_l^*(i, i)\}$. By the assumption of the theorem, there exist $j_1 \neq j_2$ such that

$$\mathcal{A} \models \bigwedge_{\langle k,l\rangle\in J} t_k^*(j_1, j_2)\ \neq\ t_l^*(j_1, j_2).$$

Note that we can effectively find the minimal pair of elements with this property because $\mathcal{A}$ is computable. Now,

1) Move every $d = t(\bar{c}) \in \cup_{i\in\omega} C_i^s$ to the set $C_k^{s,1}$, where $k = t^*(j_1, j_2)$. In particular, note that a_i is moved to $C_{j_1}^{s,1}$ and b_i is moved to $C_{j_2}^{s,1}$. Let n_{s+1} be the maximal i such that $C_i^{s,1} \neq \varnothing$.
2) For each $n_s < i \leqslant n_{s+1}$, put new elements a_i, b_i, e_i into $C_i^{s,2}$ and let $C_i^{s,2} = C_i^{s,1}$ for $i \leqslant n_s$.
3) For every $i \leqslant r$, every tuple $a_1, \ldots, a_{m_i} \in \{0, \ldots, n_{s+1}\}$ such that $f_i(\bar{a}) \leqslant n_{s+1}$, and every tuple $c_1, \ldots, c_{m_i}$ such that $c_j \in C_{a_j}^{s,2}$, if $h_i(\bar{c})$ has not yet been defined, then add a new element to $C_{f_i(\bar{a})}^{s,3}$ and declare it to be the value of $h_i(\bar{c})$.

Let $C_i^{s+1} = C_i^{s,3}$ for all $i \leqslant n_{s+1}$. This concludes step $s + 1$.

The following lemmas describe some properties of the construction.

Lemma 5.4.2. *For all s and every $c, d \in \cup_{i\in\omega} C_i^s$, if $g_s(c) \neq g_s(d)$, then $g_{s+1}(c) \neq g_{s+1}(d)$.*

Proof. Let $\widetilde{c} = t_1(\bar{c}_1)$ and $\widetilde{d} = t_2(\bar{c}_2)$. If we do not split any pair $\{a_i, b_i\}$ at step $s+1$, then, clearly, $g_{s+1}(c) = g_s(c) \neq g_s(d) = g_{s+1}(d)$. Suppose that we split $\{a_i, b_i\}$ at step $s+1$. Consider the terms $t_1^*(x, y)$, $t_2^*(x, y)$. Then $g_{s+1}(c) = t_1^*(j_1, j_2)$ and $g_{s+1}(d) = t_2^*(j_1, j_2)$. Since $t_1^*(i, i) = g_s(t_1(\bar{c}_1)) = g_s(c) \neq g_s(d) = g_s(t_2(\bar{c}_2)) = t_2^*(i, i)$ and we choose $j_1 \neq j_2$ such that they preserve the inequality, we have $g_{s+1}(c) \neq g_{s+1}(d)$.

□

Lemma 5.4.3. *For all s, $n_s < n_{s+1}$.*

Proof. If we do not split any pair $\{a_i, b_i\}$ at step $s+1$, then $n_{s+1} = n_s + 1$. Suppose that we split some $\{a_i, b_i\}$ at this step. For each $j \leqslant n_s$, consider the ground element $e_j \in C_j^s$. Also consider the ground elements a_i, b_i from C_i^s. By our construction, $e_j \in C_j^{s+1}$ for all $j \leqslant n_s$, and $a_i \in C_{j_1}^{s+1}, b_i \in C_{j_2}^{s+1}$. If j_1 or j_2 is less than or equal to n_s, then it equals i. Since $j_1 \neq j_2$, it is impossible that $j_1, j_2 \leqslant n_s$. Hence, $j_1 > n_s$ or $j_2 > n_s$ and, therefore, $n_{s+1} > n_s$.

□

Lemma 5.4.4. *For every $i \leqslant r$ and every m_i-tuple $\bar{c}$, there exists a step s at which $h_i(\bar{c})$ is defined. Hence h_i is a total computable function.*

Proof. Take some s_0 such that $\bar{c} \in \cup C_i^{s_0}$. Let $\bar{c} = c_1, \dots, c_{m_i}$ and consider the terms $\widetilde{c_j} = t_j(\bar{d}_j)$, $j \leqslant m_i$. Take the minimal n such that all the tuples $\bar{d}_j$, $j \leqslant m_i$, of ground elements belong to the set $\{a_0, b_0, e_0, \dots, a_n, b_n, e_n\}$. Take $s_1 \geqslant s_0$ such that after step s_1 we do not split any pair $\{a_i, b_i\}$, $i \leqslant n$. This means that for all $s \geqslant s_1$, $g_s(c_j) = g_{s_1}(c_j)$. Let $g_{s_1}(c_j) = a_j$ and take $s_2 \geqslant s_1$ such that $f_i(\bar{a}) \leqslant n_{s_2}$. Such s_2 exists by Lemma 5.4.3. Now, if $h_i(\bar{c})$ has not yet been defined, then, since $c_j \in C_{a_j}^{s_2}$ and $f_i(\bar{a}) \leqslant n_{s_2}$, we will define $h_i(\bar{c})$ at this step.

□

Now take any $d \in \mathbb{N}$ and consider the term $\widetilde{d} = t(\bar{c})$. There exists a step s_0 after which we do not split any pair $\{a_i, b_i\}$ of ground elements such that $a_i \in \bar{c}$ or $b_i \in \bar{c}$. Then $g_s(d) = g_{s_0}(d)$ for all $s \geqslant s_0$. This means that there exists $g(d) = \lim_s g_s(d)$. Let $C_i = \{d : g(d) = i\}$. Note that $C_i \neq \varnothing$ because $e_i \in C_i$.

Lemma 5.4.5. *At every step s, the following properties hold:*

(i) for every $i \leqslant r$ and every m_i-tuples $\bar{c}^1$ and $\bar{c}^2$ such that $g_s(\bar{c}^1) = g_s(\bar{c}^2)$, if $h_i(\bar{c}^1)$ and $h_i(\bar{c}^2)$ are both defined, then $g_s(h_i(\bar{c}^1)) = g_s(h_i(\bar{c}^2))$,

(ii) $\psi_s : i \to C_i^s$ is a partial isomorphism between $\mathcal{A} \cap \{0, \dots, n_s\}$ and $\{C_i^s\}_{i \leqslant n_s}$.

Proof. First, note that (ii) implies (i). Now prove (ii) by induction on s. It suffices to prove the following statement:

> for every $i \leqslant r$, every m_i-tuple $\bar{a}$ and every m_i-tuple $\bar{c}$ such that $c_j \in C_{a_j}^{s,1}$, if $h_i(\bar{c})$ is defined, then $f_i(\bar{a}) \leqslant n_{s+1}$ and $h_i(\bar{c}) \in C_{f_i(\bar{a})}^{s,1}$.

This is because when we put new elements into $C_i^{s,2}$ or $C_i^{s,3}$, we do it according to the partial isomorphism.

If we do not split any pair of ground elements at step $s+1$, then there is nothing to prove. Suppose that we split $\{a_i, b_i\}$ at this step. Then we move every d such that $\widetilde{d} = t(\bar{c})$ to the set $C_k^{s,1}$, where $k = t^*(j_1, j_2)$.

Take any m_i-tuple $\bar{c}$ such that $c_j \in C_{a_j}^{s,1}$ and $h_i(\bar{c})$ is defined. Let $\widetilde{c_j} = t_j(\bar{u}_j)$. Then, by the construction, $a_j = t_j^*(j_1, j_2)$. So, we have

$$g_{s+1}(h_i(\bar{c})) = g_{s+1}(h_i(t_1(\bar{u}_1), \dots, t_{m_i}(\bar{u}_{m_i}))) = \\ f_i(t_1^*(j_1, j_2), \dots, t_{m_i}^*(j_1, j_2)) = f_i(\bar{a}).$$

Also note that $f_i(\bar{a}) \leqslant n_{s+1}$ by the choice of n_{s+1}.

□

Consider a relation η defined as follows:

$$\langle x, y\rangle \in \eta \Longleftrightarrow g(x) = g(y).$$

Lemma 5.4.6. *η is a congruence relation on $(\mathbb{N}, h_1, \dots, h_r)$ and $(\mathbb{N}, h_1, \dots, h_r)/\eta$ is isomorphic to $\mathcal{A}$.*

Proof. Obviously, η is an equivalence relation. Now, take any h_i and two m_i-tuples $\bar{c}^1$ and $\bar{c}^2$ such that $g(\bar{c}^1) = g(\bar{c}^2)$. Take s_0 such that $h_i(\bar{c}^1)$ and $h_i(\bar{c}^2)$ are defined at step s_0 and

$$\forall s \geqslant s_0 \quad g_s(\bar{c}^1) = g(\bar{c}^1) \ \& \ g_s(\bar{c}^2) = g(\bar{c}^2) \ \& \\ g_s(h_i(\bar{c}^1)) = g(h_i(\bar{c}^1)) \ \& \ g_s(h_i(\bar{c}^2)) = g(h_i(\bar{c}^2)).$$

From Lemma 5.4.5(i) it follows that $\forall s \geqslant s_0 \ \ g_s(h_i(\bar{c}^1)) = g_s(h_i(\bar{c}^2))$ and, therefore, $g(h_i(\bar{c}^1)) = g(h_i(\bar{c}^2))$. So, η is a congruence.

Recall that $C_i = \{d : g(d) = i\}$. Now prove that the mapping $\psi : i \to C_i$ gives us an isomorphism between $\mathcal{A}$ and $(\mathbb{N}, h_1, \dots, h_r)/\eta$. Take any m_i-tuple $\bar{a}$ and m_i-tuple $\bar{c}$ such that $g(\bar{c}) = \bar{a}$. We need to prove that $g(h_i(\bar{c})) = f_i(\bar{a})$.

Take s_0 such that $h_i(\bar{c})$ is defined at the step s_0 and

$$\forall s \geqslant s_0 \quad g_s(\bar{c}) = g(\bar{c}) \text{ and } g_s(h_i(\bar{c})) = g(h_i(\bar{c})).$$

From Lemma 5.4.5(ii) it follows that $g_s(h_i(\bar{c})) = f_i(\bar{a})$ for all $s \geqslant s_0$. Hence $g(h_i(\bar{c})) = f_i(\bar{a})$.

□

Lemma 5.4.7. *η is a Π_1^0 relation whose Turing degree is* **d**.

Proof. Show that $\mathbb{N}^2 \setminus \eta$ is Σ_1^0. We have

$$\langle x, y\rangle \notin \eta \Longleftrightarrow g(x) \neq g(y) \Longleftrightarrow \exists s\, (x, y \in \cup_{i\in\omega} C_i^s \ \& \ g_s(x) \neq g_s(y)),$$

where the second equivalence follows from Lemma 5.4.2. Therefore, η is Π_1^0.

Now prove that the degree of η is $\mathbf{d}$. From the construction of Theorem 5.4.1 it follows that $i \in D$ iff $\langle a_i, b_i \rangle \notin \eta$. Hence $D \leqslant_T \eta$. Show that $\eta \leqslant_T D$. Take any two numbers x, y and find the least s such that $x, y \in \cup_{i \leqslant n_s} C_i^s$. Let $\widetilde{x} = t_1(\bar{d}_1)$ and $\widetilde{y} = t_2(\bar{d}_2)$, where $\bar{d}_1, \bar{d}_2 \in \{a_0, b_0, c_0, \ldots, a_{n_s}, b_{n_s}, c_{n_s}\}$. Find the least $s_1 \geqslant s$ such that we have split all the pairs $\{a_i, b_i\}$ for $i \in D \cap \{0, \ldots, n_s\}$ by step s_1. Then $\langle x, y \rangle \in \eta$ iff $g_{s_1}(x) = g_{s_1}(y)$.

□

Therefore, the theorem is proved.

□

This theorem together with the Proposition 5.3.3 give us the following examples of computable algebras that admit non-computable Π_1^0-presentations.

Corollary 5.4.8. *The following algebras possess non-computable* Π_1^0*-presentations:*

1) arithmetic $(\omega, S, +, \times)$,

2) term algebras,

3) infinite computable fields $(F, +, \times, 0, 1)$,

4) computable torsion-free abelian groups,

5) infinite computable vector spaces over a finite field.

Chapter 6
Finite automata presentable abelian groups

In this chapter we construct new examples of FA presentable torsion-free abelian groups and also provide a new FA presentation of the group $\mathbb{Z} \times \mathbb{Z}$. Here is the outline of the chapter.

In the first section we will give the precise definition of an FA presentable structure that was informally discussed in the General Introduction. In the second section we describe an FA presentation of the group R_p that will be used in our main construction. In Section 6.3 we introduce the notion of an amalgamated product for monoids and abelian groups. In the next section we prove that under certain conditions the amalgamated product of FA presentable groups or monoids is itself FA presentable. In Section 6.5 we describe indecomposable and strongly indecomposable abelian groups and show that they are FA presentable using the methods from Section 6.4. Finally, in the last section of this chapter we construct a new automatic presentation of the group $\mathbb{Z} \times \mathbb{Z}$ such that every nontrivial cyclic subgroup in that presentation is not FA recognizable.

6.1 Preliminaries

We now give the formal definitions that will be used in this chapter.

Definition 6.1.1. Let Σ be a finite alphabet, and $\bar{a} = (a_1, \ldots, a_k)$ be a tuple of words from Σ^*. A *convolution* of $\bar{a}$ is a word in the alphabet $(\Sigma \cup \{\Box\})^k$ which is constructed by placing the words $a_1, \ldots, a_k$ one under another and adding a special symbol $\Box$ at the end of some words to get the same length. For example,

$$\mathrm{Conv}(01, 1011, 100) = \begin{matrix} 0 & 1 & \Box & \Box \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & \Box \end{matrix}$$

A *convolution of a relation* $R \subseteq (\Sigma^*)^k$ is defined as $\mathrm{Conv}(R) = \{\mathrm{Conv}(\bar{a}) : \bar{a} \in R\}$.

Definition 6.1.2. A relation $R \subseteq (\Sigma^*)^k$ is *FA recognizable*, or *regular*, if $\mathrm{Conv}(R)$ is recognized by a finite automaton.

Definition 6.1.3. A structure $\mathcal{A} = (A; R_1, \ldots, R_n, f_1, \ldots, f_m)$ is *FA presented* if, for a finite alphabet Σ, $A \subseteq \Sigma^*$ is an FA recognizable set of words in Σ^*, and all the relations $R_1, \ldots, R_n$ together with the graphs of the operations $f_1, \ldots, f_m$ are recognized by finite automata.

A structure $\mathcal{A}$ is *FA presentable* if it is isomorphic to an FA presented structure.

In some cases in order to prove that a given structure is FA presentable we will not construct its automatic presentation explicitly. Instead, we will give its first-order interpretation in a structure already known to be FA presented. The description of this method together with the formal definitions and proofs can be found in [4].

6.2 An FA presentation of the group R_p

Definition 6.2.1. Let p be either a prime number or a product of different primes. Then R_p is the subgroup of $(\mathbb{Q}, +)$ consisting of the elements of the form k/p^i.

In the literature R_p is also denoted by $\mathbb{Q}^{(p)}$ (for example, see [16]) or $\mathbb{Z}[1/p]$. The next theorem shows that R_p is FA presentable, and we will use this particular presentation of R_p in Section 6.5 to construct new examples of FA presentable abelian groups.

Theorem 6.2.2. *R_p is FA-presentable.*

Proof. First, we will construct an automatic presentation of R_p^+, the submonoid of R_p consisting of the elements greater than or equal to 0. Later we describe how to obtain an FA presentation of the entire group R_p from the one for R_p^+.

The alphabet of the FA presentation of R_p^+ will be $\Sigma = \{\binom{n}{m} \,:\, n \in \{0,1\}$ and $m \in \{0, \ldots, p-1\}\}$. Every element $z \in R_p^+$ will be represented by two lines of digits,

n_1	n_2	$\cdots$	n_k
m_1	m_2	$\cdots$	m_k

where $n_1 n_2 \ldots n_k$ represents the integral part of z in binary presentation with the least significant digit first, and $m_1 m_2 \ldots m_k$ represents the fractional part of z in base p with the most significant digit first. If needed, we put additional zeros to the right to make the lengths of the integral and fractional parts to be equal. For example, if $p = 3$, then the element $14\frac{17}{27} \in R_3^+$ is represented by

0	1	1	1
1	2	2	0

Let the domain D of the FA presentation of R_p^+ consist of all words in Σ^* not ending in $\binom{0}{0}$ except for $\binom{0}{0}$ itself, which represents 0. Clearly, D is FA recognizable.

Let *Add* be the graph of the addition operation. We prove that *Add* is FA recognizable. First, we construct an auxiliary automaton $\mathcal{A}$ whose alphabet is $(\Sigma \cup \{\square\})^3$. The states of $\mathcal{A}$ are q_0, $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, where q_0 is the initial state and $(0,0)$ is the final state. The state (α, β) denotes the fact that we have a carry bit α in the addition of the integral parts and a carry bit β in the addition of the fractional parts.

The transitions of $\mathcal{A}$ are defined below. It is assumed there that the special symbol $\Box$ is identical to the symbol $\binom{0}{0}$.

There is a transition from q_0 to (α, β) with the label $\left(\binom{n_1}{m_1}, \binom{n_2}{m_2}, \binom{n_3}{m_3}\right)^\top$ if and only if

$$\begin{cases} n_1 + n_2 = 2\alpha + n_3 \\ m_1 + m_2 + \beta = m_3 \end{cases} \quad \text{or} \quad \begin{cases} n_1 + n_2 + 1 = 2\alpha + n_3 \\ m_1 + m_2 + \beta = p + m_3 \ . \end{cases}$$

This means that from the first letter of the input $\mathcal{A}$ guesses the carry bit from the fractional part to the integral part: in the first case the carry bit is 0, while in the second case the carry bit is 1.

There is a transition from (α, β) to (α', β') with the label $\left(\binom{n_1}{m_1}, \binom{n_2}{m_2}, \binom{n_3}{m_3}\right)^\top$ if and only if

$$\begin{cases} n_1 + n_2 + \alpha = 2\alpha' + n_3 \\ m_1 + m_2 + \beta' = p\beta + m_3 \ . \end{cases}$$

Now, as one can see, $\mathrm{Conv}(\mathit{Add}) = L(\mathcal{A}) \cap \mathrm{Conv}(D^3)$. Therefore, since D^3 is FA recognizable, then so is *Add*.

Let us define an FA presentation of R_p. Consider the presentation of R_p^+ given above; let $\pi : D^2 \to D^2$ be the following function

$$\pi(x, y) = \begin{cases} (x - y, 0) & \text{if } x \geqslant y, \\ (0, y - x) & \text{if } x < y. \end{cases}$$

Note that the graph of π is an FA recognizable subset of D^4 since it can be defined in terms of *Add* and $\leqslant$ relations which are FA recognizable in our presentation of R_p^+. Now the domain of the FA presentation of R_p is

$$\big\{(x, y) : \ x, y \in D \text{ and } (x = 0 \vee y = 0)\big\}$$

with the addition operation defined as

$$(x_1, y_1) + (x_2, y_2) = (x_3, y_3) \text{ if and only if } (x_3, y_3) = \pi(x_1 + x_2, y_1 + y_2).$$

$\Box$

6.3 Amalgamations of monoids and abelian groups

Before turning to abelian groups, let us consider commutative monoids which have the cancellation property, namely, $a + c = b + c$ implies $a = b$ for all elements a, b, c. In what follows, by a *monoid* we will mean a *commutative monoid with cancellation property.*

Proposition 6.3.1. *Let M, N, and U be monoids, and $f : U \to M$, $g : U \to N$ be isomorphic embeddings. Consider the direct product $M \times N$ of the monoids and a relation $\sim_U$ on $M \times N$ defined as follows:*

$$(x_0, y_0) \sim_U (x_1, y_1) \iff \exists u, v \in U \; \big(x_0 + f(u) = x_1 + f(v) \; \wedge \\ y_0 + g(u) = y_1 + g(v) \big)$$

Then $\sim_U$ is a congruence on $M \times N$, and $M \oplus_U N$, the **amalgamated product** *of M and N over U, is the quotient structure $M \times N / \sim_U$, which is also a commutative monoid with cancellation property.*

Proof. It is straightforward to show that $\sim_U$ is a congruence and that $M \oplus_U N$ is a commutative monoid. We prove that it possesses the cancellation property. Suppose $(x_0, y_0) + (z, w) \sim_U (x_1, y_1) + (z, w)$; then $x_0 + z + f(u) = x_1 + z + f(v)$ and $y_0 + w + g(u) = y_1 + w + g(v)$ for some $u, v \in U$. Since M and N possess the cancellation property, we have that $x_0 + f(u) = x_1 + f(v)$ and $y_0 + g(u) = y_1 + g(v)$, that is, $(x_0, y_0) \sim_U (x_1, y_1)$.

□

We will use the notation $\langle x, y\rangle_U$ to denote the equivalence class of $(x, y) \in M \times N$ with respect to $\sim_U$.

Proposition 6.3.2. *Let $M \oplus_U N$ be an amalgamated product of monoids M and N over U. Then there are submonoids $\widetilde{M}$ and $\widetilde{N}$ in $M \oplus_U N$ such that $\widetilde{M} \cong M$, $\widetilde{N} \cong N$, and $M \oplus_U N = \widetilde{M} + \widetilde{N}$.*

Proof. Let $\widetilde{M} = \{\langle x, 0\rangle_U : x \in M\}$ and $\widetilde{N} = \{\langle 0, y\rangle_U : y \in N\}$; as one can see, $\widetilde{M}$ and $\widetilde{N}$ are submonoids of $M \oplus_U N$, and $M \oplus_U N = \widetilde{M} + \widetilde{N}$. Consider the mappings $\varphi : M \to \widetilde{M}$ and $\psi : N \to \widetilde{N}$ such that $\varphi(x) = \langle x, 0\rangle_U$ and $\varphi(y) = \langle 0, y\rangle_U$. Clearly, φ and ψ are epimorphisms. Let us show, for instance, that φ is one-to-one. Suppose $\langle x, 0\rangle_U = \langle x', 0\rangle_U$; then $x + f(u) = x' + f(v)$ and $g(u) = g(v)$ for some $u, v \in U$. Therefore, $u = v$ and the cancellation property implies that $x = x'$.

□

In the case of abelian groups we can define the notion of an amalgamated product in a slightly different manner.

Definition 6.3.3. Let A, B, and U be abelian groups and $f : U \to A$, $g : U \to B$ be isomorphic embeddings. Then $A \oplus_U B$, the *amalgamated product* of A and B over U, is the quotient group $A \oplus B/\widetilde{U}$ where $\widetilde{U} = \{(f(u), g(u)) \mid u \in U\}$.

The next proposition is the strengthening of 6.3.2 for abelian groups.

Proposition 6.3.4. *Let $A \oplus_U B$ be an amalgamated product of A and B over U. Then there are subgroups $\widetilde{A}$ and $\widetilde{B}$ in $A \oplus_U B$ such that $\widetilde{A} \cong A$, $\widetilde{B} \cong B$, $\widetilde{A} \cap \widetilde{B} \cong U$, and $A \oplus_U B = \widetilde{A} + \widetilde{B}$, where $\widetilde{A} + \widetilde{B} = \{a + b \mid a \in \widetilde{A},\ b \in \widetilde{B}\}$.*

Proof. By definition $A \oplus_U B = A \oplus B/\widetilde{U}$. Let $\widetilde{A} = \{(a, 0) + \widetilde{U} \mid a \in A\}$ and $\widetilde{B} = \{(0, b) + \widetilde{U} \mid b \in B\}$. As one can see, $A \oplus_U B = \widetilde{A} + \widetilde{B}$ and $\widetilde{A} \cong A$, $\widetilde{B} \cong B$. We now prove that $\widetilde{A} \cap \widetilde{B} \cong U$. Let $x \in \widetilde{A} \cap \widetilde{B}$, then $x = (a, 0) + \widetilde{U}$ and $x = (0, b) + \widetilde{U}$; hence $(a, -b) \in \widetilde{U}$ and $a = f(u)$, $b = -g(u)$. Therefore, $x = (f(u), 0) + \widetilde{U}$ and $\widetilde{A} \cap \widetilde{B} = \{(f(u), 0) + \widetilde{U} \mid u \in U\}$ which is isomorphic to U.

□

Remark 6.3.5. The intersection $\widetilde{M} \cap \widetilde{N}$ of the submonoids of $M \oplus_U N$ defined in the proof of Proposition 6.3.2 is not necessarily isomorphic to U. To show this, let M, N, U be $(\mathbb{N}, +)$ and f, g be the identity embeddings. As one can see, $M \oplus_U N$ is isomorphic to $(\mathbb{Z}, +)$ because $\langle x, y\rangle_{\mathbb{N}} = \langle x', y'\rangle_{\mathbb{N}}$ iff $x - y = x' - y'$, and we can identify $\langle x, y\rangle_{\mathbb{N}}$ with $x - y \in \mathbb{Z}$. In this case, $\widetilde{M}$ and $\widetilde{N}$ correspond to the submonoids of non-negative and non-positive numbers, respectively. Thus $\widetilde{M} \cap \widetilde{N} = \{\langle 0, 0\rangle_{\mathbb{N}}\} \ncong \mathbb{N}$.

The converse of 6.3.4 also holds.

Proposition 6.3.6. *Let L be an abelian group, A, B be subgroups of L, and $U = A \cap B$. Then*

$$A + B \cong A \oplus_U B,$$

where the embeddings f, g of U into A and B are the identity mappings.

Proof. In this case, $A \oplus_U B = A \oplus B/\widetilde{U}$, where $\widetilde{U} = \{(u, u) \mid u \in U\}$. Let $\varphi : A \oplus_U B \to A + B$ be defined as follows:

$$\varphi((a, b) + \widetilde{U}) = a - b.$$

We show that φ is an isomorphism. First, note that it is well defined: if $(a, b) + \widetilde{U} = (a', b') + \widetilde{U}$, then $(a - a', b - b') = (u, u)$ for some $u \in U$; therefore, $a - b = (a' + u) - (b' + u) = a' - b'$.

It is easy to see that φ is an epimorphism. We now prove that it is one-to-one. Let $a - b = a' - b'$; then $a - a' = b - b' \in A \cap B = U$. Therefore, $(a - a', b - b') = (u, u) \in \widetilde{U}$ and $(a, b) + \widetilde{U} = (a', b') + \widetilde{U}$.

□

Remark 6.3.7. If M, N, and U are abelian groups, then both definitions of an amalgamated product, that is the one for the groups and the one for the monoids, give us the same structure $M \oplus_U N$.

6.4 Constructions of FA presentable monoids and abelian groups

In this section we will prove a version of Proposition 6.3.2 for FA presentable structures.

Theorem 6.4.1. *If M, N, and U are FA presented monoids and $f : U \to M$, $g : U \to N$ are isomorphic embeddings that are FA recognizable subsets of $U \times M$ and $U \times N$, respectively, then the amalgamated product $M \oplus_U N$ is FA presentable. Moreover, $M \oplus_U N$ contains FA recognizable submonoids $\widetilde{M}$ and $\widetilde{N}$ such that $\widetilde{M} \cong M$, $\widetilde{N} \cong M$, and $M \oplus_U N = \widetilde{M} + \widetilde{N}$.*

Proof. We prove that $M \oplus_U N$ is FA presentable by constructing its interpretation in the FA presentable structure $E = M \sqcup N \sqcup U$ enriched with the unary predicates for the subsets M, N, U and the binary predicates R_f and R_g for the graphs of f and g. Let R^M and R^N be the graphs of the addition operation in M and N, respectively.

The domain of $M \oplus_U N$ is defined in E^2 by the formula $\Delta(x_0, y_0) = M(x_0) \wedge N(y_0)$. Addition is defined by

$$\Phi(x_0, y_0, x_1, y_1, x_2, y_2) = R^M(x_0, x_1, x_2) \;\wedge\; R^N(y_0, y_1, y_2).$$

Equality is defined by

$$\begin{aligned}\epsilon(x_0, y_0, x_1, y_1) = \exists u, v\ (U(u) \wedge U(v) \wedge x_0 + f(u) = x_1 + f(v)\\ \wedge\, y_0 + g(u) = y_1 + g(v))\end{aligned}$$

or more formally

$$\begin{aligned}\epsilon(x_0, y_0, x_1, y_1) =& \exists u, v, w_0, w_1, w_2, w_3, z_0, z_1\ (U(u) \wedge U(v) \wedge R_f(u, w_0)\\ &\wedge R_f(v, w_1) \wedge R_g(u, w_2) \wedge R_g(v, w_3) \wedge R^M(x_0, w_0, z_0)\\ &\wedge R^M(x_1, w_1, z_0) \wedge R^N(y_0, w_2, z_1) \wedge R^N(y_1, w_3, z_1)).\end{aligned}$$

From the proof of Proposition 6.3.2 it follows that $\widetilde{M}$ and $\widetilde{N}$ are defined by the formulas

$$\begin{aligned}(z_0, z_1) \in \widetilde{M} \iff& \exists x, u, v\ (M(x) \wedge U(u) \wedge U(v) \wedge\\ & z_0 + f(u) = x + f(v) \wedge z_1 + g(u) = g(v)),\\ (z_0, z_1) \in \widetilde{N} \iff& \exists y, u, v\ (N(y) \wedge U(u) \wedge U(v) \wedge\\ & z_0 + f(u) = f(v) \wedge z_1 + g(u) = y + g(v)).\end{aligned}$$

Therefore, $\widetilde{M}$ and $\widetilde{N}$ are FA recognizable submonoids.

□

Theorem 6.4.2. *Let A and B be abelian groups such that B is a subgroup of A and $|A : B|$ is finite. If B is FA presentable, then so is A.*

Proof. Let $r_0, \ldots, r_k$ be representatives of the cosets of B in A. Then there are a function $g : \{0, \ldots, k\}^2 \to \{0, \ldots, k\}$ and elements $b_{ij} \in B$ with the following property: for every i and j,

$$r_i + r_j = r_{g(i,j)} + b_{ij}.$$

We may assume that the FA presentation of B uses an alphabet Σ, such that $0, \ldots, k \notin \Sigma$, and that the domain of this presentation is $D \subseteq \Sigma^*$. Let the alphabet of the FA presentation of A be $\Sigma \cup \{0, \ldots, k\}$. Each element of A has the unique form $r_i + b$ for some $b \in B$ and is represented by the string iv, where $v \in D$ represents b. Since A is abelian,

$$(r_i + b_1) + (r_j + b_2) = r_{g(i,j)} + b_{ij} + b_1 + b_2.$$

Hence, the graph of the addition operation can be recognized by a finite automaton. □

Example 6.4.3 (Two different presentations of R_6)**.** Consider the presentation of R_6 described in Section 6.2. We will show that R_6 in this presentation does not have an FA recognizable subgroup isomorphic to R_2. Suppose M is an FA recognizable subgroup of R_6 and $M \cong R_2$. Let $M^+ = \{(x, 0) : (x, 0) \in M\}$; then M^+ is FA recognizable and $M^+ \cong R_2^+$. Note that we can identify the FA presentation of R_6^+ and the FA recognizable submonoid of R_6 with the domain $\{(x, 0) : (x, 0) \in R_6\}$. This implies that R_6^+ has an FA recognizable submonoid isomorphic to R_2^+.

Now if $M^+ \leqslant R_6^+$ is isomorphic to R_2^+, then for some $n_0, k_0 \in \mathbb{N}$

$$M^+ = \frac{n_0}{6^{k_0}} \cdot R_2^+ = \left\{ \frac{n_0 n 3^k}{6^{k_0+k}} \ : \ k, n \in \mathbb{N} \right\}.$$

For each k, let α_k be the smallest element of M^+ of length k_0+k in this presentation. Obviously, $\alpha_k = n_0 3^k 6^{-(k_0+k)}$ and it has the form

0	0	$\cdots$	0	0	$\cdots$	0
0	0	$\cdots$	0	r_k		

where

$$\lim_{k \to \infty} \frac{\mathit{length}(r_k)}{(k + k_0)} = \log_6 3. \tag{6.1}$$

Choosing sufficiently large k we will have enough leading zeros in the presentation of α_k to pump this string. This will give us a contradiction with the formula (6.1). Therefore, M^+ is not FA recognizable, and M is not FA recognizable too.

On the other hand, R_6 is isomorphic to $R_2^+ \oplus_{\mathbb{N}} R_3^+$. Indeed, $R_2^+ \oplus_{\mathbb{N}} R_3^+ = \{\langle x, y \rangle_{\mathbb{N}} : (x, y) \in R_2^+ \times R_3^+\}$ and

$$\langle x, y \rangle_{\mathbb{N}} = \langle x', y' \rangle_{\mathbb{N}} \text{ if and only if } x - y = x' - y'.$$

Let $z = m/6^k \in R_6$; there are $m_0, m_1 \in \mathbb{Z}$ such that $m = 3^k m_0 - 2^k m_1$; then $z = m_0/2^k - m_1/3^k = (m_0/2^k + l) - (m_1/3^k + l)$ for any $l \in \mathbb{Z}$. Choosing sufficiently large l, we see that $z = x - y$, where $x \in R_2^+$, $y \in R_3^+$. Therefore, the mapping that sends $\langle x, y \rangle_{\mathbb{N}}$ to $x - y \in R_6$ gives us desired isomorphism.

Consider the FA presentations of R_2^+ and R_3^+ described in Section 6.2. Recall that the integral part of every element is presented in base 2 both in R_2^+ and R_3^+. Thus, if we take the FA presentation of $\mathbb{N}$ in base 2, then the graphs of the identity embeddings $f : \mathbb{N} \to R_2^+$ and $g : \mathbb{N} \to R_3^+$ will be FA recognizable. Therefore, by Theorem 6.4.1, R_6 has an FA presentation which contains FA recognizable submonoids isomorphic to R_2^+ and R_3^+. Now if $M^+ \subseteq R_6$ is a submonoid isomorphic to R_2^+, then $M = M^+ \cup -M^+$ is a subgroup isomorphic to R_2. Clearly, M is definable in terms of M^+ and addition. Therefore, this presentation of R_6 contains FA recognizable subgroups isomorphic to R_2 and R_3. It is different from the presentation given in Section 6.2 in the sense that there is no automatic isomorphism between them.

6.5 Indecomposable FA presentable abelian groups

We describe rank n torsion-free abelian groups, G_n and H_n, which are indecomposable and strongly indecomposable, respectively. We then show how to apply the methods from the previous section to prove that they are FA presentable.

In what follows, we will use an expression like $p^{-\infty}a$ as an abbreviation for the infinite set $p^{-1}a, p^{-2}a, \ldots$. For $n \geqslant 2$, let G_n be the subgroup of $\mathbb{Q}^n$ generated by $p_1^{-\infty}e_1, \ldots, p_n^{-\infty}e_n$, $q^{-1}(e_1 + \cdots + e_n)$, where $q, p_1, \ldots, p_n$ are different primes and $e_1, \ldots, e_n$ are linear independent elements in $\mathbb{Q}^n$ considered as a $\mathbb{Q}$-vector space. An example of such group can be found in [16, vol. 2, §88, Exercise 2].

Definition 6.5.1. A torsion-free abelian group A is *indecomposable* if for all B and C, $A = B \oplus C$ implies $B = \mathbf{0}$ or $C = \mathbf{0}$.

Theorem 6.5.2. *The group G_n is indecomposable for any $n \geqslant 2$.*

Proof. First, note that every $x \in G_n$ has the form

$$x = (p_1^{-k_1}m_1 + q^{-1}s)e_1 + \cdots + (p_n^{-k_n}m_n + q^{-1}s)e_n,$$

where $m_1, \ldots, m_n, s \in \mathbb{Z}$ and $k_1, \ldots, k_n \in \mathbb{N}$. Let $E_j = \langle p_j^{-\infty}e_j \rangle$ where $1 \leqslant j \leqslant n$. We show that the groups E_j are fully invariant in G_n, i.e. $\varphi(E_j) \subseteq E_j$ for any endomorphism φ of G_n. Let $x \in E_j$ and $\varphi(x) = \sum s_i e_i$. In G_n, x is divisible by all the powers of p_j, and so is $\varphi(x)$. Hence, $s_i = 0$ for $i \neq j$ and $\varphi(x) = s_j e_j$.

Take any $i \neq j$. As mentioned above, s_j has the form $p_j^{-k_j}m_j + q^{-1}s$ and s_i has the form $p_i^{-k_i}m_i + q^{-1}s$. Since $s_i = 0$, $q^{-1}s$ must be an integer. Therefore, $\varphi(x) = s_j e_j$ belongs to E_j.

Now suppose that $G_n = A \oplus B$. If $x \in G_n$, then x has the unique form $x = a + b$, where $a \in A$, $b \in B$. Define the following endomorphisms of G_n: $\varphi^A(x) = a$ and

$\varphi^B(x) = b$, where $x = a + b$. Obviously, $x = \varphi^A(x) + \varphi^B(x)$. If $x \in E_j$, then $\varphi^A(x) \in E_j \cap A$ and $\varphi^B(x) \in E_j \cap B$ since E_j is fully invariant. This means that $E_j = (E_j \cap A) \oplus (E_j \cap B)$.

Note that E_j is indecomposable because it has rank 1. Therefore, $E_j \subseteq A$ or $E_j \subseteq B$. Assume there exists $1 \leqslant k < n$ such that, possibly after re-indexing, $E_1, \dots, E_k \subseteq A$ and $E_{k+1}, \dots, E_n \subseteq B$. Let $q^{-1}(e_1 + \cdots + e_n) = a + b$, where $a \in A$ and $b \in B$. Then $e_1 + \cdots + e_k + e_{k+1} + \cdots + e_n = qa + qb$. Since $e_1 + \cdots + e_k \in A$ and $e_{k+1} + \cdots + e_n \in B$, we have that $a = q^{-1}(e_1 + \cdots + e_k)$.

We show that this is impossible. Let $a = (p_1^{-k_1}m_1 + q^{-1}s)e_1 + \cdots + (p_n^{-k_n}m_n + q^{-1}s)e_n$; since $p_n^{-k_n}m_n + q^{-1}s = 0$, $q^{-1}s$ must be an integer. Hence $p_1^{-k_1}m_1 + q^{-1}s$ cannot be equal to q^{-1}.

So, we can assume that $E_1, \dots, E_n \subseteq A$. If $B \neq \mathbf{0}$, then let $b \in B$ be a nonzero element. Then there exists $m > 0$ such that $mb \in \langle e_1, \dots, e_n \rangle \subseteq E_1 + \cdots + E_n \subseteq A$, which is impossible because $mb \neq 0$ and mb is an element of B. Therefore, $B = \mathbf{0}$. $\square$

Definition 6.5.3 ([2])**.** A torsion-free abelian group A is *strongly indecomposable* if whenever $0 \neq k \in \mathbb{N}$ and $kA \leqslant B \oplus C \leqslant A$, then $B = \mathbf{0}$ or $C = \mathbf{0}$.

The group H_n from the next theorem was introduced in [2, Example 2.4].

Theorem 6.5.4. *The group* $H_n = gr(p_1^{-\infty}e_1, \dots, p_n^{-\infty}e_n,\ q^{-\infty}(e_1 + \cdots + e_n))$ *is strongly indecomposable for any* $n \geqslant 2$*.*

Proof. First, we show that any endomorphism of H_n is the same as the multiplication by an integer. Let $x \in H_n$; by an argument similar to one at the beginning of the proof of Theorem 6.5.2, one can show that if x is divisible in H_n by all the powers of p_i, then x has the form $mp_i^{-k}e_i$.

Now let $\widehat{e}_1 = -e_1, \dots, \widehat{e}_{n-1} = -e_{n-1}$, $\widehat{e}_n = e_1 + \cdots + e_n$. Then $e_n = \widehat{e}_1 + \cdots + \widehat{e}_n$ and we can write

$$H_n = gr(p_1^{-\infty}\widehat{e}_1, \dots, p_{n-1}^{-\infty}\widehat{e}_{n-1},\ p_n^{-\infty}(\widehat{e}_1 + \cdots + \widehat{e}_n),\ q^{-\infty}\widehat{e}_n)).$$

Therefore, if x is divisible in H_n by any power of q, then it has the form $mq^{-k}\widehat{e}_n = mq^{-k}(e_1 + \cdots + e_n)$.

Let φ be an endomorphism of H_n; then $\varphi(e_i) = r_ie_i$, where $r_i = m_ip_i^{-k_i}$, because $\varphi(e_i)$ is divisible by any power of p_i. Hence $\varphi(e_1 + \cdots + e_n) = \sum r_ie_i$. On the other hand, since $\varphi(e_1 + \cdots + e_n)$ is divisible by all the powers of q, it has the form $mq^{-k}(e_1 + \cdots + e_n)$. Therefore, each r_i is equal to an integer number r and $\varphi(x) = rx$. Since the group is torsion-free, every nonzero endomorphism is one-to-one.

To conclude the proof, we will show that if a torsion-free abelian group A has only one-to-one nonzero endomorphisms, then it is strongly indecomposable. Assume that there are $k \neq 0$ and nonzero groups B and C such that $kA \leqslant B \oplus C \leqslant A$. Let ψ be an endomorphism of $B \oplus C$ defined as follows: if $x = b + c$, where $b \in B$, $c \in C$, then $\psi(x) = b$. Then the mapping φ defined by $\varphi(x) = \psi(kx)$

is an endomorphism of A. Take any $0 \neq c \in C$, then $\varphi(c) = \psi(kc) = 0$ and, therefore, φ is not one-to-one. Note that φ is also nonzero since if $0 \neq b \in B$, then $\varphi(b) = kb \neq 0$.

□

Theorem 6.5.5. *The group G_n is FA presentable.*

Proof. Since, by Theorem 6.2.2, R_p is FA presentable, the direct sum $R_{p_1} \oplus \cdots \oplus R_{p_n}$ is also FA presentable. Note that $R_{p_1} \oplus \cdots \oplus R_{p_n}$ is a subgroup of finite index in G_n. Hence, by Theorem 6.4.2, G_n is also FA presentable.

□

Remark 6.5.6. Note that, unlike G_n, the group H_n is not an extension of finite index of any known example of an FA presentable group. To show that it is FA presentable we will use the method of amalgamated products described in Section 6.4.

Theorem 6.5.7. *The group H_n is FA presentable.*

Proof. First, let us show that H_n is isomorphic to $(R^+_{p_1} \times \cdots \times R^+_{p_n}) \oplus_{\mathbb{N}} R^+_q$, an amalgamated product of the monoids $R^+_{p_1} \times \cdots \times R^+_{p_n}$ and R^+_q over $\mathbb{N}$, where the isomorphic embeddings $f : \mathbb{N} \to R^+_{p_1} \times \cdots \times R^+_{p_n}$ and $g : \mathbb{N} \to R^+_q$ are chosen as follows: for all $m \in \mathbb{N}$, $f(m) = (m, \ldots, m)$ and $g(m) = m$. Note that our proof will show that the amalgamated product of these monoids is actually a group.

Every element of $(R^+_{p_1} \times \cdots \times R^+_{p_n}) \oplus_{\mathbb{N}} R^+_q$ is of the form $\langle (a_1, \ldots, a_n), b\rangle_{\mathbb{N}}$, where $a_i \in R^+_{p_i}$, for $i = 1, \ldots, n$, and $b \in R^+_q$. Suppose that

$$\langle (a_1, \ldots, a_n), b\rangle_{\mathbb{N}} = \langle (a'_1, \ldots, a'_n), b'\rangle_{\mathbb{N}}.$$

Then there are $u, v \in \mathbb{N}$ such that

$$\begin{cases} (a_1 + u, \ldots, a_n + u) = (a'_1 + v, \ldots, a'_n + v) \\ b + u = b' + v. \end{cases}$$

This implies that $a_i - b = a'_i - b'$ for all $i = 1, \ldots, n$. Thus we can correctly define a function h on $(R^+_{p_1} \times \cdots \times R^+_{p_n}) \oplus_{\mathbb{N}} R^+_q$ such that

$$h(\langle (a_1, \ldots, a_n), b\rangle_{\mathbb{N}}) = (a_1 - b)e_1 + \cdots + (a_n - b)e_n.$$

As one can see, the range of h is a subset of H_n, and h is a homomorphism. To show that it is one-to-one, assume that

$$h(\langle (a_1, \ldots, a_n), b\rangle_{\mathbb{N}}) = h(\langle (a'_1, \ldots, a'_n), b'\rangle_{\mathbb{N}});$$

then

$$(a_1 - b)e_1 + \cdots + (a_n - b)e_n = (a'_1 - b')e_1 + \cdots + (a'_n - b')e_n.$$

Therefore, $a_i - a_i' = b - b' \in R_{p_i} \cap R_q$ for $i = 1, \ldots, n$. Since $R_{p_i} \cap R_q = \mathbb{Z}$, there is $w \in \mathbb{Z}$ such that

$$\begin{cases} (a_1, \ldots, a_n) = (a_1' + w, \ldots, a_n' + w) \\ b = b' + w. \end{cases}$$

So, $\langle (a_1, \ldots, a_n), b \rangle_{\mathbb{N}} = \langle (a_1', \ldots, a_n'), b' \rangle_{\mathbb{N}}$.

Now, to prove that h is onto, consider an element $z \in H_n$; it must be of the form

$$z = \Big(\frac{m_1}{p_1^{k_1}} + \frac{l}{q^r}\Big)e_1 + \cdots + \Big(\frac{m_n}{p_n^{k_n}} + \frac{l}{q^r}\Big)e_n$$

for some integers m_i, l and natural numbers k_i, r. Obviously,

$$\frac{m_i}{p_i^{k_i}} + \frac{l}{q^r} = \Big(\frac{m_i}{p_i^{k_i}} + t\Big) - \Big(-\frac{l}{q^r} + t\Big)$$

for any $t \in \mathbb{Z}$. Choosing sufficiently large t, we can make all $a_i = m_i/p_i^{k_i} + t$ and $b = -l/q^r + t$ to be positive. In this case $\langle (a_1, \ldots, a_n), b \rangle_{\mathbb{N}}$ is an element of $(R^+_{p_1} \times \cdots \times R^+_{p_n}) \oplus_{\mathbb{N}} R^+_q$ and $h(\langle (a_1, \ldots, a_n), b \rangle_{\mathbb{N}}) = z$. Therefore, the range of h is H_n, and hence it is an isomorphism.

Consider the FA presentations of the monoids $R^+_{p_1}, \ldots, R^+_{p_n}$, and R^+_q described in Section 6.2. From this we can easily construct an FA presentation of $R^+_{p_1} \times \cdots \times R^+_{p_n}$ by putting the strings representing the elements of $R^+_{p_i}$'s one under another in columns using an extra padding symbol when necessary. Recall that the integral part of an element of $R^+_{p_i}$ or R^+_q is presented in base 2. Therefore, if we consider the presentation of $\mathbb{N}$ in base 2, then the graphs of the isomorphic embeddings $f : \mathbb{N} \to R^+_{p_1} \times \cdots \times R^+_{p_n}$ and $g : \mathbb{N} \to R^+_q$ will be FA recognizable. Now, by Theorem 6.4.1, the structure $(R^+_{p_1} \times \cdots \times R^+_{p_n}) \oplus_{\mathbb{N}} R^+_q$ is FA presentable, and, as shown above, it is isomorphic to H_n.

□

6.6 A new FA presentation of $\mathbb{Z} \times \mathbb{Z}$

Let $(\mathbb{Z}, +)$ be the group of integers under addition. In this section we will construct an FA presentation of $(\mathbb{Z}, +)^2$ in which no nontrivial cyclic subgroup is FA recognizable.

Consider $\mathbb{Z}[x]/\langle p_3 \rangle$, the quotient of the polynomial ring $\mathbb{Z}[x]$ with respect to the ideal generated by $p_3(x) = x^2 + x - 3$. We will use the notation $p(x) \sim q(x)$ to denote that $p_3(x)$ divides $p(x) - q(x)$.

Remark 6.6.1. In the construction described below we can use any polynomial of the form $x^2 + x - q$, for a prime $q \geqslant 3$, instead of $p_3(x)$.

Let $A = (\mathbb{Z}[x]/\langle p_3\rangle, +)$ be the additive group of the ring $\mathbb{Z}[x]/\langle p_3\rangle$. It is not hard to see that A is isomorphic to $\mathbb{Z}^2$ since every polynomial in $\mathbb{Z}[x]$ is equivalent over $\langle p_3\rangle$ to a linear polynomial $kx + l$, which can be identified with a pair $(k, l) \in \mathbb{Z}^2$.

We say that a polynomial $a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ is in *reduced form* (or briefly *reduced*) if $|a_i| \leqslant 2$ for all $i \leqslant n$.

Proposition 6.6.2. *For every $p(x) \in \mathbb{Z}[x]$, there is a reduced polynomial $\widetilde{p}(x)$ equivalent to it.*

Proof. This can be proved by induction: assume that $p(x)$ is in reduced form and show that $p(x) \pm x^n$ is equivalent to a reduced polynomial. It is enough to consider the case $p(x) \pm 1$ since $p(x) \pm x^n$ can be rewritten as $(q(x) \pm 1)x^n + r(x)$, where $q(x)$ and $r(x)$ are in reduced form and $\deg(r) < n$. Note that if $p_0(x)$ and $p_1(x)$ are in reduced forms and have non-negative coefficients, then $p_0(x) - p_1(x)$ is reduced. Moreover, any reduced $p(x)$ is equal to the difference of such $p_0(x)$ and $p_1(x)$. So, it is enough to consider the case when $p(x)$ is reduced and has non-negative coefficients and to show that $p(x) + 1$ is equivalent to a reduced polynomial with non-negative coefficients.

We will actually prove a stronger statement: if $p(x)$ is a reduced polynomial with non-negative coefficients, then $p(x) + (a_1 x + a_0)$, where $0 \leqslant a_0, a_1 \leqslant 2$, is equivalent to a polynomial of the same sort. The proof is now by induction on the degree of $p(x)$.

Let us write $p(x)$ as $p(x) = p_1(x)x^2 + (b_1 x + b_0)$. Now, using the fact that $3 \sim x^2 + x$, we have

$$\begin{aligned} p(x) + (a_1 x + a_0) &= p_1(x)x^2 + (a_1 + b_1)x + (a_0 + b_0) \\ &\sim (p_1(x) + r_1 x + (r_0 + r_1))x^2 + c_1 x + c_0, \end{aligned}$$

where

$$c_0 = \begin{cases} a_0 + b_0, & \text{if } a_0 + b_0 < 3 \\ a_0 + b_0 - 3, & \text{otherwise} \end{cases} \qquad r_0 = \left[\frac{a_0 + b_0}{3}\right],$$

$$c_1 = \begin{cases} a_1 + b_1 + r_0, & \text{if } a_1 + b_1 + r_0 < 3 \\ a_1 + b_1 + r_0 - 3, & \text{otherwise} \end{cases} \qquad r_1 = \left[\frac{a_1 + b_1 + r_0}{3}\right].$$

Here, $[v]$ is the integral part of v defined by

$$[v] = \begin{cases} \max\{k \in \mathbb{Z} : k \leqslant v\} & \text{if } v \geqslant 0, \\ \min\{k \in \mathbb{Z} : v \leqslant k\} & \text{if } v < 0. \end{cases}$$

For example, $[1.5] = 1$ and $[-1.5] = -1$. Note that $0 \leqslant c_0, c_1 \leqslant 2$ and $0 \leqslant r_0, r_1 \leqslant 1$. By induction, $p_1(x) + r_1 x + (r_0 + r_1)$ is equivalent to a reduced polynomial with non-negative coefficients; hence so is $p(x)$.

□

We now describe an automatic presentation of the group A. The alphabet of the presentation is $\Sigma = \{-2, -1, 0, 1, 2\}$. Each reduced polynomial of the form $a_n x^n + \cdots + a_0$ is represented by the word $a_0 \ldots a_n \in \Sigma^*$. We say that two words $a_0 \ldots a_n$ and $b_0 \ldots b_m$ in Σ^* are *equivalent* if $a_n x^n + \cdots + a_0 \sim b_m x^m + \cdots + b_0$.

This equivalence relation is FA recognizable. An algorithm for checking it is as follows. Given two words $a_0 \ldots a_n$ and $b_0 \ldots b_m$, we can assume that $n = m$ since one can always add extra zeros to the right. The algorithm needs to remember two carries r_0, r_1; initially $r_0 = r_1 = 0$. Note that since $3 \sim x + x^2$, whenever we subtract 3 from any digit we need to add 1 to the next two digits in order to get an equivalent word. That is why we need two carries here.

Now, for every $i = 0, \ldots, n$, do the following. Check if 3 divides $a_i - b_i + r_0$. If 'no', then the words are not equivalent. If 'yes', then let $r_0^{\text{old}} = r_0$, $r_1^{\text{old}} = r_1$; redefine

$$r_0 = r_1^{\text{old}} + \frac{a_i - b_i + r_0^{\text{old}}}{3}, \qquad r_1 = \frac{a_i - b_i + r_0^{\text{old}}}{3}$$

and go to the step $i + 1$. If we reach in this way the nth step, then the words are equivalent if and only if $a_n - b_n + r_0 = 0$ and $r_1 = 0$.

Since at every step $|r_0| \leqslant 4$ and $|r_1| \leqslant 2$, this algorithm requires a constant amount of memory. Now it is not hard to construct a finite automaton recognizing the equivalence.

Consider the following order on Σ: $-2 < -1 < 0 < 1 < 2$. It naturally extends to the length-lexicographical order on Σ^*, denoted as $<_{llex}$. Let the domain of the FA presentation of A be

$$\text{Dom}(A) = \{w \in \Sigma^* : (\forall u <_{llex} w)\ u \text{ is not equivalent to } w\}.$$

This set is FA recognizable since $<_{llex}$ is an FA recognizable relation.

To define addition on $\text{Dom}(A)$, consider the relation $R(x, y, z)$ such that if $x = a_0 \ldots a_k$, $y = b_0 \ldots b_l$, then $z = c_0 \ldots c_n$ is obtained from x and y by applying the following algorithm. Again, let r_0, r_1 be two carries that are initially zero. For every step i starting from 0, let c_i be such that $|c_i| < 3$,

$$c_i \equiv a_i + b_i + r_0 \pmod 3,$$

and c_i has the same sign as $a_i + b_i + r_0$. Let $r_0^{\text{old}} = r_0$, $r_1^{\text{old}} = r_1$. Now redefine

$$r_0 = r_1^{\text{old}} + \left[\frac{a_i + b_i + r_0^{\text{old}}}{3}\right], \qquad r_1 = \left[\frac{a_i + b_i + r_0^{\text{old}}}{3}\right]$$

and go to the step $i + 1$.

For example, if $x = 2211$ and $y = 22$ then this algorithm produces $z = 120021$. By construction, if $R(x, y, z)$ holds, then the polynomial corresponding to z is equivalent over $\langle p_3 \rangle$ to the sum of the polynomials represented by x and y. It is easy to see that at every step, $|r_0| \leqslant 4$ and $|r_1| \leqslant 2$. Thus, as before, R can be recognized by a finite automaton.

Let $Add(x, y, z)$ be defined as

$$Add = \{(x, y, z) : x, y, z \in \mathrm{Dom}(A) \text{ and } \\ \exists w\ (R(x, y, w)\ \wedge\ w \text{ is equivalent to } z)\}.$$

Since $\mathrm{Dom}(A)$, R, and the equivalence relation are FA recognizable, Add is also FA recognizable. Obviously, Add is the graph of the addition operation on $\mathrm{Dom}(A)$, and the FA presented structure $(\mathrm{Dom}(A), Add)$ is isomorphic to A.

Our next goal is to show that no nontrivial cyclic subgroup in this presentation of $\mathbb{Z}^2$ is FA recognizable.

Lemma 6.6.3. *Let $p(x)$ and $q(x)$ be reduced polynomials such that $p(x) \sim q(x)$ and $x^k \mid p(x)$. Then $x^k \mid q(x)$.*

Proof. Suppose that $x^k \nmid q(x)$; then $p(x) - q(x) = x^l(a_0 + a_1x + \cdots)$, where $l < k$, $|a_0| \leqslant 2$, and $a_0 \neq 0$. Since $3 \nmid a_0$, $p_3(x) = x^2 + x - 3$ cannot divide $p(x) - q(x)$, which gives a contradiction.

□

For $p(x) \in \mathbb{Z}[x]$, consider the set of words in Σ^* that represent the polynomials equivalent to $p(x)$. All these words start with the same number of zeros. So, we say that $p(x)$ *starts with k zeros in reduced form* if there is $w \in \Sigma^*$ representing $p(x)$ that starts with k zeros.

The following lemma will be used several times later on.

Lemma 6.6.4. *Let n be an integer; then $3^k \mid n$ if and only if n starts with k zeros in reduced form.*

Proof. Suppose that $3^k \mid n$; then $n = 3^k m \sim x^k(x+1)^k m$. Taking a reduced form for $(x+1)^k m$ and multiplying it by x^k, we obtain a reduced form for n that starts with k zeros. Thus n starts with k zeros in reduced form.

The other implication can be proved by induction on k. First, suppose that n starts with one 0 in reduced form and $n = 3m + r$, where $0 < r \leqslant 2$. Take any reduced form for $3m$. Since it starts with 0, $n = 3m + r$ has a reduced form that starts with $r \neq 0$. This contradicts our assumption, and hence $3 \mid n$.

It is not hard to see that if $p(x)$ starts with exactly k zeros in reduced form and $q(x)$ starts with exactly l zeros, then any reduced form for $p(x)q(x)$ starts with exactly $k + l$ zeros. Now suppose that n starts with $k + 1$ zeros in reduced form; then $n = 3m$ and m starts with k zeros because $3 \sim 011$ starts with one 0. By induction, $3^k \mid m$, and so we have $3^{k+1} \mid n$.

□

Let $\alpha = (\sqrt{13} - 1)/2$ be the positive root of $p_3(x) = x^2 + x - 3$. Consider the mapping $F : \mathbb{Z}[x] \to \mathbb{R}$ defined as $F : p(x) \mapsto p(\alpha)$. Obviously,

$$(p + q)(\alpha) = p(\alpha) + q(\alpha) \quad \text{and} \quad (pq)(\alpha) = p(\alpha)q(\alpha).$$

Furthermore, if $p(x) \sim q(x)$, then $p(\alpha) = q(\alpha)$ since $p(\alpha) - q(\alpha) = p_3(\alpha)r(\alpha) = 0$.

Consider an arbitrary nontrivial cyclic subgroup in our presentation of $\mathbb{Z}^2$. It has the form

$$\langle w \rangle = \{n \cdot w : \; n \in \mathbb{Z}\}$$

for some $w \in \mathrm{Dom}(A)$. Let $q(x)$ be the polynomial that corresponds to w. Note that $q(\alpha) \neq 0$ since w represents nonzero element. Indeed, by applying the Euclidean algorithm one can see that there are polynomials s, r with $\deg(r) < 2$ such that $q = s \cdot p_3 + r$. Moreover, since the leading coefficient of p_3 is 1, s and r have integer coefficients. Now if $q(\alpha) = 0$, then $r(\alpha) = 0$, which implies that $r = 0$ since α is irrational. So, q is equal to $s \cdot p_3$ and hence is equivalent to 0. This contradicts our assumption that w represents nonzero element.

Suppose that $\langle w \rangle$ is recognized by a finite automaton with d states. We know that $3^d \cdot w \in \langle w \rangle$ starts with d zeros in reduced form. So, let $3^d \cdot w$ be equivalent to $0^d v \in \mathrm{Dom}(A)$. By the Pumping Lemma, there are $s_0, s_1, t \in \mathbb{N}$ with $t \neq 0$ such that $0^d v = 0^{s_0} 0^t 0^{s_1} v$ and $w_k = 0^{tk+s_0+s_1} v \in \langle w \rangle$ for all $k \geqslant 0$.

Let $q_k(x)$ be the polynomial that corresponds to w_k. Since $w_k \in \langle w \rangle$, we have that $w_k \sim n_k \cdot w$ for some $n_k \in \mathbb{Z}$. If w starts with m zeros, then n_k starts with at least $tk - m + s_0 + s_1$ zeros in reduced form; thus $3^{tk-m+s_0+s_1} \mid n_k$.

The fact that w_k is equivalent to $n_k \cdot w$ implies that $q_k(\alpha) = n_k q(\alpha)$. Now, on the one hand,

$$\begin{aligned}|q_k(\alpha)| &\leqslant 2(1 + \alpha + \cdots + \alpha^{|w_k|-1}) \\ &= 2\frac{\alpha^{|w_k|} - 1}{\alpha - 1} \leqslant 8\alpha^{|w_k|} = 8\alpha^{s_0+s_1+|v|}\alpha^{tk} = C_0\alpha^{tk}.\end{aligned}$$

On the other hand,

$$|q_k(\alpha)| = |n_k||q(\alpha)| \geqslant 3^{tk} 3^{s_0+s_1-m}|q(\alpha)| = C_1 3^{tk}.$$

Therefore,

$$C_1 3^{tk} \leqslant C_0 \alpha^{tk} \quad \text{for all } k \geqslant 0,$$

which is impossible because $\alpha < 3$. Hence $\langle w \rangle$ is not FA recognizable.

BIBLIOGRAPHY

[1] S. Akiyama, C. Frougny, and J. Sakarovitch. Powers of rationals modulo 1 and rational base systems. *Preprint*, 2005. 14

[2] D. M. Arnold. *Finite rank torsion free abelian groups and rings*, volume 931 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982. 69

[3] J. T. Baldwin and A. H. Lachlan. On strongly minimal sets. *J. Symbolic Logic*, 36:79–96, 1971. 3, 4

[4] A. Blumensath and E. Grädel. Automatic structures. In *15th Annual IEEE Symposium on Logic in Computer Science (Santa Barbara, CA, 2000)*, pages 51–62. IEEE Comput. Soc. Press, Los Alamitos, CA, 2000. 62

[5] C. S. Calude. *Information and randomness. An algorithmic perspective*. Texts in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, second edition, 2002. 23

[6] P. Cholak, S. Goncharov, B. Khoussainov, and R. A. Shore. Computably categorical structures and expansions by constants. *J. Symbolic Logic*, 64(1):13–37, 1999. 9

[7] P. Cholak, R. A. Shore, and R. Solomon. A computably stable structure with no Scott family of finitary formulas. *Arch. Math. Logic*, 45(5):519–538, 2006. 6

[8] P. A. Cholak, S. Lempp, M. Lerman, and R. A. Shore, editors. *Computability theory and its applications. Current trends and open problems*, volume 257 of *Contemporary Mathematics*, Providence, RI, 2000. American Mathematical Society. 2

[9] C. Delhommé. Non-automaticity of ω^ω. *Manuscript*, 2001. 13

[10] J. L. Ershov. *Decision problems and constructive models*. Nauka, Moscow, 1980 (Russian). 1

[11] Y. L. Ershov and S. S. Goncharov. *Constructive models*. Siberian School of Algebra and Logic. Consultants Bureau, New York, 2000. 1

[12] Y. L. Ershov, S. S. Goncharov, A. Nerode, J. B. Remmel, and V. W. Marek, editors. *Handbook of recursive mathematics. Vol. 1 and Vol. 2*, volume 138–

139 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1998. 2

[13] L. Feiner. Hierarchies of Boolean algebras. *J. Symbolic Logic*, 35:365–374, 1970. 10

[14] A. Fröhlich and J. C. Shepherdson. On the factorisation of polynomials in a finite number of steps. *Math. Z.*, 62:331–334, 1955. 1

[15] A. Fröhlich and J. C. Shepherdson. Effective procedures in field theory. *Philos. Trans. Roy. Soc. London. Ser. A.*, 248:407–432, 1956. 1

[16] L. Fuchs. *Infinite abelian groups. Vol. I and Vol. II.* Pure and Applied Mathematics, Vol. 36 and Vol. 36-II. Academic Press, New York, 1970, 1973. 62, 68

[17] S. Goncharov and B. Khoussainov. Open problems in the theory of constructive algebraic systems. In *Computability theory and its applications (Boulder, CO, 1999)*, volume 257 of *Contemp. Math.*, pages 145–170. Amer. Math. Soc., Providence, RI, 2000. 2

[18] S. Goncharov and B. Khoussainov. Complexity of theories of computable categorical models. *Algebra Logika*, 43(6):650–665, 758–759, 2004. 4

[19] S. S. Goncharov. Selfstability and computable families of constructivizations. *Algebra i Logika*, 14(6):647–680, 727, 1975. 1

[20] S. S. Goncharov. Constructive models of $\aleph_1$-categorical theories. *Mat. Zametki*, 23(6):885–888, 1978. 1, 4, 5

[21] S. S. Goncharov. Computable univalent numerations. *Algebra i Logika*, 19(5):507–551, 617, 1980. 1, 8, 9, 35

[22] S. S. Goncharov. The problem of the number of nonautoequivalent constructivizations. *Algebra i Logika*, 19(6):621–639, 745, 1980. 1, 8

[23] S. S. Goncharov, V. S. Harizanov, M. C. Laskowski, S. Lempp, and C. F. D. McCoy. Trivial, strongly minimal theories are model complete after naming constants. *Proc. Amer. Math. Soc.*, 131(12):3901–3912 (electronic), 2003. 4, 5

[24] S. S. Goncharov and A. T. Nurtazin. Constructive models of complete decidable theories. *Algebra i Logika*, 12:125–142, 243, 1973. 2

[25] E. R. Griffor, editor. *Handbook of computability theory*, volume 140 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999. 2

[26] V. S. Harizanov. The possible Turing degree of the nonzero member in a two element degree spectrum. *Ann. Pure Appl. Logic*, 60(1):1–30, 1993. 9

[27] L. Harrington. Recursively presentable prime models. *J. Symbolic Logic*, 39:305–309, 1974. 2, 3, 5

[28] B. Herwig, S. Lempp, and M. Ziegler. Constructive models of uncountably categorical theories. *Proc. Amer. Math. Soc.*, 127(12):3711–3719, 1999. 5

[29] D. R. Hirschfeldt. *Degree Spectra of Relations on Computable Structures*. PhD Thesis, Cornell University, 1999. 9

[30] D. R. Hirschfeldt, B. Khoussainov, and P. Semukhin. An uncountably categorical theory whose only computably presentable model is saturated. *Notre Dame J. Formal Logic*, 47(1):63–71 (electronic), 2006. 5, 33

[31] D. R. Hirschfeldt, B. Khoussainov, and R. A. Shore. A computably categorical structure whose expansion by a constant has infinite computable dimension. *J. Symbolic Logic*, 68(4):1199–1241, 2003. 9

[32] D. R. Hirschfeldt, B. Khoussainov, R. A. Shore, and A. M. Slinko. Degree spectra and computable dimensions in algebraic structures. *Ann. Pure Appl. Logic*, 115(1-3):71–113, 2002. 10, 35, 39, 47

[33] W. Hodges. *Model theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993. 6, 25, 31

[34] W. Hodges. *A shorter model theory*. Cambridge University Press, Cambridge, 1997. 6

[35] B. R. Hodgson. *Théories décidables par automate fini*. PhD Thesis, University of Montreal, 1976. 12

[36] B. R. Hodgson. Décidabilité par automate fini. *Ann. Sci. Math. Québec*, 7(1):39–57, 1983. 12

[37] N. K. Kasymov. Algebras with residually finite positively presented expansions. *Algebra i Logika*, 26(6):715–730, 771, 1987. 10

[38] N. K. Kasymov. Positive algebras with congruences of finite index. *Algebra i Logika*, 30(3):293–305, 380, 1991. 10

[39] N. K. Kasymov and B. M. Khusainov. Finitely generated enumerable and absolutely locally finite algebras. *Vychisl. Sistemy*, (116, Prikladn. Logika):3–15, 162, 1986. 10

[40] N. G. Khisamiev. On strongly constructive models of decidable theories. *Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.*, (1):83–84, 1974. 3, 5

[41] N. G. Khisamiev. Criterion for constructivizability of a direct sum of cyclic p-groups. *Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.*, (1):51–55, 86, 1981. 15

[42] N. G. Khisamiev. Theory of abelian groups with constructive models. *Sibirsk. Mat. Zh.*, 27(4):128–143, 215, 1986. 15

[43] N. G. Khisamiev. Constructive abelian groups. In *Handbook of recursive mathematics, Vol. 2*, volume 139 of *Stud. Logic Found. Math.*, pages 1177–1231. North-Holland, Amsterdam, 1998. 15

[44] B. Khoussainov, S. Lempp, and T. A. Slaman. Computably enumerable algebras, their expansions, and isomorphisms. *Internat. J. Algebra Comput.*, 15(3):437–454, 2005. 10, 11, 52

[45] B. Khoussainov and A. Nerode. Automatic presentations of structures. In *Logic and computational complexity (Indianapolis, IN, 1994)*, volume 960 of *Lecture Notes in Comput. Sci.*, pages 367–392. Springer, Berlin, 1995. 12

[46] B. Khoussainov, A. Nies, S. Rubin, and F. Stephan. Automatic structures: richness and limitations. *Log. Methods Comput. Sci.*, 3(2):2:2, 18 pp. (electronic), 2007. 13

[47] B. Khoussainov, A. Nies, and R. A. Shore. Computable models of theories with few models. *Notre Dame J. Formal Logic*, 38(2):165–178, 1997. 4, 5, 8, 15, 17, 18, 22, 33

[48] B. Khoussainov and S. Rubin. Automatic structures: overview and future

directions. *J. Autom. Lang. Comb.*, 8(2):287–301, 2003. Weighted automata: theory and applications (Dresden, 2002). 13

[49] B. Khoussainov, P. Semukhin, and F. Stephan. Applications of kolmogorov complexity to computable model theory. *J. Symbolic Logic*, 72(3):1041–1054, 2007. 8

[50] B. Khoussainov and R. A. Shore. Computable isomorphisms, degree spectra of relations, and Scott families. Computability theory. *Ann. Pure Appl. Logic*, 93, 98(1-3):153–193, 297–298, 1998, 1999. 6, 9

[51] B. Khoussainov and R. A. Shore. Effective model theory: the number of models and their complexity. In *Models and computability (Leeds, 1997)*, volume 259 of *London Math. Soc. Lecture Note Ser.*, pages 193–239. Cambridge Univ. Press, Cambridge, 1999. 2

[52] B. Khoussainov, T. Slaman, and P. Semukhin. Π_1^0-presentations of algebras. *Arch. Math. Logic*, 45(6):769–781, 2006. 12

[53] B. M. Khoussainov, M. C. Laskowski, S. Lempp, and R. Solomon. On the computability-theoretic complexity of trivial, strongly minimal models. *Proc. Amer. Math. Soc.*, 135(11):3711–3721, 2007. 4

[54] K. Ž. Kudaĭbergenov. Constructivizable models of undecidable theories. *Sibirsk. Mat. Zh.*, 21(5):155–158, 192, 1980. 4, 5

[55] O. V. Kudinov. An autostable 1-decidable model without a computable Scott family of ∃-formulas. *Algebra i Logika*, 35(4):458–467, 498, 1996. 6

[56] M. Li and P. Vitányi. *An introduction to Kolmogorov complexity and its applications*. Texts and Monographs in Computer Science. Springer-Verlag, New York, 1993. 23

[57] J. Love. Stability among r.e. quotient algebras. *Ann. Pure Appl. Logic*, 59(1):55–63, 1993. 10

[58] A. I. Mal'cev. Constructive algebras. I. *Uspehi Mat. Nauk*, 16(3 (99)):3–60, 1961. 1

[59] D. Marker. *Model theory. An introduction*, volume 217 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002. 6, 31

[60] G. Metakides and A. Nerode. Recursion theory and algebra. In *Algebra and logic (Fourteenth Summer Res. Inst., Austral. Math. Soc., Monash Univ., Clayton, 1974)*, pages 209–219. Lecture Notes in Math., Vol. 450. Springer, Berlin, 1975. 1

[61] T. S. Millar. Foundations of recursive model theory. *Ann. Math. Logic*, 13(1):45–72, 1978. 2

[62] M. Morley. Categoricity in power. *Trans. Amer. Math. Soc.*, 114:514–538, 1965. 3

[63] M. Morley. Decidable models. *Israel J. Math.*, 25(3-4):233–240, 1976. 2

[64] A. Nies. A new spectrum of recursive models. *Notre Dame J. Formal Logic*, 40(3):307–314, 1999. 4, 5

[65] A. Nies and P. Semukhin. Finite automata presentable abelian groups. In *Symposium on Logical Foundations of Computer Science (New York, June 4–7, 2007)*, volume 4514 of *Lecture Notes in Comput. Sci.*, pages 422–436. Springer-Verlag, Berlin, 2007. 14

[66] A. Nies and R. M. Thomas. Finite automaton presentable groups and rings. *J. Algebra, to appear.* 14

[67] P. Odifreddi. *Classical recursion theory*, volume 125 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1989. 23

[68] G. P. Oliver and R. M. Thomas. Automatic presentations for finitely generated groups. In *STACS 2005*, volume 3404 of *Lecture Notes in Comput. Sci.*, pages 693–704. Springer, Berlin, 2005. 14

[69] M. O. Rabin. Recursive unsolvability of group theoretic problems. *Ann. of Math. (2)*, 67:172–194, 1958. 1

[70] M. O. Rabin. Computable algebra, general theory and theory of computable fields. *Trans. Amer. Math. Soc.*, 95:341–360, 1960. 1

[71] R. I. Soare. *Recursively enumerable sets and degrees*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987. 23, 49

[72] B. L. van der Waerden. Eine bemerkung über die unzerlegbarkeit von polynomen. *Mathematische Annalen*, 102:738–739, 1930. 1

[73] B. L. van der Waerden. *Moderne Algebra I.* Julius Springer, Berlin, first edition, 1930. 1