Prime models of finite computable dimension

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Abstract

We study the following open question in computable model theory: does there exist a structure of computable dimension two which is the prime model of its first-order theory? We construct an example of such a structure by coding a certain family of c.e. sets with exactly two one-to-one computable enumerations into a directed graph. We also show that there are examples of such structures in the classes of undirected graphs, partial orders, lattices, and integral domains.

1 Introduction

Computable Model Theory studies the effective content of typical mathematical concepts, constructions, and theorems, especially from algebra and classical model theory. One of the most fundamental notions here is that of an *isomorphism*. In algebra and model theory we usually identify isomorphic structures and consider them to be the same. However, when studying computable models, one can see that isomorphic structures might have different computability-theoretic properties. We, therefore, introduce the notion of a *computable isomorphism*, instead of the classical one, and use it as a tool to distinguish two different computable presentations of the same structure. This approach leads us to the notion of a *computable dimension* which is defined below.

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Definition 1.1. A structure $\mathcal{A}$ is *computable* if its domain $A \subseteq \omega$ and its atomic diagram are computable. $\mathcal{A}$ is *computably presentable* if it is isomorphic to a computable structure, which is called a *computable presentation* of $\mathcal{A}$.

Definition 1.2. The *computable dimension* of a structure $\mathcal{A}$ is the number of its computable presentations up to computable isomorphism. If the computable dimension of $\mathcal{A}$ is 1 we say that $\mathcal{A}$ is *computably categorical.*

In this paper we will answer the following open question in computable model theory: does there exist a structure of computable dimension two which is the prime model of its own theory? It is easy to give examples of prime models of dimension 1 or ω. For instance, the countable dense linear order without endpoints and one successor structure $(\mathbb{N}, S)$ are computably categorical, while $(\mathbb{N}, \leqslant)$ and $\mathcal{B}_\omega$, the Boolean algebra of finite and cofinite subsets of the naturals, have infinite computable dimensions.

However it is much more difficult to construct a structure of finite computable dimension $k > 1$. Goncharov [2, 3] was the first to give an example of such structure. In [2] he constructed a uniformly computably enumerable (u.c.e.) family $\mathcal{F}$ of sets that has exactly two non-equivalent one-to-one computable enumerations. This family is then encoded into a computable graph G in such a way that the computable dimension of G is equal to the number of non-equivalent one-to-one computable enumerations of $\mathcal{F}$.

Since then many improvements to the construction have been made to obtain various strengthenings of this result. For example, Cholak/Goncharov/Khoussainov/Shore [1] showed that for each $k > 1$, there is a computably categorical structure $\mathcal{A}$ such that any expansion of $\mathcal{A}$ by a single constant has computable dimension k. This construction was further improved by Hirschfeldt/Khoussainov/Shore [6] who showed that it is possible to make the dimension of the expanded structure infinite.

The research on the structures of finite computable dimension is also related to the study of degree spectra of relations on computable models. The *degree spectrum* of a relation R on a computable structure $\mathcal{A}$ is the set of Turing degrees of images of R in all computable presentations of $\mathcal{A}$. Harizanov [4] showed that there exists a relation U in a structure $\mathcal{A}$ of computable dimension two such that $\mathrm{DgSp}_{\mathcal{A}}(U) = \{\mathbf{0}, \mathbf{d}\}$, where $\mathbf{d} \leqslant \mathbf{0}'$ and does not contain a c.e. set. Later on, Khoussainov and Shore [8] proved that there exists a relation U in a structure $\mathcal{A}$ of dimension two such that $\mathrm{DgSp}_{\mathcal{A}}(U) = \{\mathbf{0}, \mathbf{d}\}$, where **d** is c.e. but not computable. Hirschfeldt [5] further improved this result by showing that **d** can be chosen to be any non-computable c.e. Turing degree.

All known examples of the structures of finite computable dimension $k > 1$ are not the prime models of their theories. Hence it was an open problem as to whether there exists a prime model of computable dimension two. This question is especially interesting because prime models are relatively simple from a model-theoretic point of view as they are elementarily embeddable into any other model of their theories. So, the problem is whether it is possible to encode enough information into a prime model to construct a structure of dimension two. The main result of this paper is the construction of such a structure.

Theorem. *There exists a structure of computable dimension two which is the prime model of its own theory.*

The construction is based on coding a u.c.e. family of sets $\mathcal{F}$ constructed by Goncharov [2] into a graph. We will use some structural properties of $\mathcal{F}$ to make the coding in such a way that every element of the graph is definable by a first order formula, which implies that the structure is prime.

We also give examples of prime models of finite computable dimension in some specific classes of algebraic structures.

Theorem. *There are prime models of computable dimension two in the following classes of structures:*

1) undirected graphs (section 3.1),

2) partially ordered sets (section 3.2),

3) lattices (section 3.3),

4) integral domains expanded by finitely many constants (section 3.4).

The construction of these examples follows the technique from Hirschfeldt/Khoussainov/Shore/Slinko [7], where they developed the methods for coding directed graphs into undirected, irreflexive graphs, partial orders, lattices, integral domains, nilpotent groups, etc. These codings preserve the following computability-theoretic properties of the structures:

(a) degree spectra of the structures;

(b) degree spectra of relations on computable structures;

(c) computable dimensions of the structures as well as computable dimensions of their expansions by a single constant.

We will show that if in the original structure $\mathcal{A}$ every element is definable by a first order formula, then the structure $\mathcal{B}$, into which we encode $\mathcal{A}$, is prime. In fact, every $b \in \mathcal{B}$ is also definable by a formula or, in the case of integral domains, there is a formula with finitely many solutions that holds on b.

The outline of the paper is as follows. In Section 2 we describe the construction of a prime directed graph of computable dimension two. Then in Section 3, using the methods from Hirschfeldt/Khoussainov/Shore/Slinko [7], we will encode this graph into an undirected graph, a partial order, a lattice, and an integral domain. It follows from [7] that all these codings preserve computable dimensions. We show that these codings also preserve the property of being the prime model. In many cases we provide an explicit proof that a given structure has computable dimension two rather then referring the reader to [7].

2 The Main Construction

The main result of this paper is the following theorem.

Theorem 2.1. *There exists a computable structure G of computable dimension two which is the prime model of its own theory.*

The structure G will be a directed graph. The proof is based on coding the u.c.e. family $\mathcal{F}$ constructed by Goncharov [2] into a computable graph of dimension two in such a way that every element of G can be defined by a first-order formula without parameters. This implies that G is the prime model of its theory. We now restate the result of S. Goncharov in more detail.

Definition 2.2. Let $\mathcal{F}$ be a u.c.e. family of sets. A *computable enumeration* $\mu : \omega \to \mathcal{F}$ is a mapping from ω onto $\mathcal{F}$ such that the set $\{(n, k) : n \in \mu(k)\}$ is c.e. We will also use the notation $\{A_i\}_{i\in\omega}$ for an enumeration μ, where $A_i = \mu(i)$.

An enumeration μ is *reducible* to ν, denoted $\mu \leqslant \nu$, if there is a computable function f such that $\mu(i) = \nu(f(i))$ for every $i \in \omega$. Two enumerations μ and ν are *equivalent*, denoted $\mu \equiv \nu$, if $\mu \leqslant \nu$ and $\nu \leqslant \mu$.

Theorem 2.3 (Goncharov [2])**.** *There exists a u.c.e. family $\mathcal{F}$ that has exactly two nonequivalent one-to-one computable enumerations. Moreover, $\mathcal{F}$ has the following properties:*

(i) *If $S \in \mathcal{F}$ is a finite set, then S contains an element $n(S)$, called a marker for a finite set S, that does not belong to* **any other** *set from $\mathcal{F}$.*

(ii) *If $S \in \mathcal{F}$ is an infinite set, then S contains an element $n(S)$, called a marker for an infinite set S, that does not belong to* **any other infinite** *set from $\mathcal{F}$.*

Remark 2.4. We may assume that the family $\mathcal{F}$ contains infinitely many one-element sets. Indeed, we can always take $\mathcal{F}' = \{2S : S \in \mathcal{F}\} \cup \{\{2k+1\} : k \in \omega\}$ instead of $\mathcal{F}$. The family $\mathcal{F}'$ has exactly two nonequivalent one-to-one computable enumerations. This follows from the fact that the index set of the subfamily $\{\{2k+1\} : k \in \omega\}$ is computable in any one-to-one computable enumeration of $\mathcal{F}'$.

Let $\{A_n^0\}_{n\in\omega}$ and $\{A_n^1\}_{n\in\omega}$ be two nonequivalent one-to-one computable enumerations of $\mathcal{F}$. For each $i = 0, 1$, fix a computable enumeration of $\{A_n^i\}_{n\in\omega}$ such that at every step s, exactly one element enters one of the A_n^i's.

We build two computable presentations G_0 and G_1 of the directed graph G using a step-by-step construction. Let G_i^s be the finite part of G_i constructed by the end of step s. When we add a new element to G_i^s, we always choose the least element that has not been used so far. At every step s, we will have that $G_i^s \subseteq G_i^{s+1}$ and $G_i = \bigcup\limits_{s\in\omega} G_i^s$. We will use the following notations in our construction.

Definition 2.5. Let $n \in \omega$; the directed graph $[n]$ has $n + 3$ many nodes $x_0, x_1, \dots, x_{n+2}$ with an edge from x_0 to itself, an edge from x_{n+2} to x_1, and an edge from x_i to x_{i+1} for $i \leqslant n + 1$. We call x_0 the *top* of $[n]$.

Let $S \subseteq \omega$; the directed graph $[S]$ consists of one copy of $[s]$ for every $s \in S$, with all tops identified.

Definition 2.6. Two tops n_0 and n_1 of G_i^s are *connected* if there is an element $l \in G_i^s$ such that (n_0, l), (l, n_0), (n_1, l), (l, n_1) are edges in G_i^s. In this case l is called the *linking element*.

"To connect two tops n_0, n_1 of G_i^s using a linking element" means to add one new element l as well as the edges (n_0, l), (l, n_0), (n_1, l), (l, n_1) to the graph G_i^s.

A *component* is a maximal subgraph isomorphic to $[S]$ for some $S \subseteq \omega$. Note that this is not necessarily the same as a maximal connected component.

The construction of G_0 and G_1 is now as follows.

Step 0. Let $G_0^0 = G_1^0 = \{2n : n \in \omega\}$ and, for every $n \in \omega$, connect $2n$ to itself in both G_0^0 and G_1^0. Thus $2n$ is a top in G_0 and G_1.

Step $s+1$. For $i \in \{0, 1\}$, do the following. Let k be the unique element that enters some A_n^i at step s. Consider the component of G_i^s isomorphic to $[A_{n,s}^i]$ and containing the top $2n$. Expand this component to one isomorphic to $[A_{n,s}^i \cup \{k\}] = [A_{n,s+1}^i]$. If k is not the first element that enters A_n^i, then find the least m such that $2n$ is not connected to $2m$ in G_i^s and connect $2n$ to $2m$ using one new linking element.

Now, for every pair u_0, v_0 of tops in G_0^s and every pair u_1, v_1 of tops in G_1^s such that u_0, u_1 belong to the components isomorphic to $[S_0]$ and v_0, v_1 belong to the ones isomorphic to $[S_1]$ for some non-empty sets S_0 and S_1, do the following. Check if u_0, v_0 are connected in G_0^s, but u_1, v_1 are not connected in G_1^s, or vice versa. If *yes*, connect those tops u_i, v_i using one linking element which are not connected in G_i^s.

End of the construction.

Lemma 2.7. *G_0 and G_1 are isomorphic.*

Proof. According to the construction, each top $2n$ in G_0 (resp. G_1) belongs to the component isomorphic to $[A_n^0]$ (resp. $[A_n^1]$). Since $\{A_n^0\}_{n\in\omega}$ and $\{A_n^1\}_{n\in\omega}$ are one-to-one enumerations of the same family, G_0 and G_1 consist of the same collection of components.

To finish the proof, we need to show that for every pair n_0, n_1 of tops in G_0 and every pair m_0, m_1 of tops in G_1, if for $i \in \{0, 1\}$, n_i and m_i belong to the isomorphic components, then n_0, n_1 are connected in G_0 iff m_0, m_1 are connected in G_1. Suppose that n_0, n_1, m_0, m_1 is a counterexample to the above statement such that, for instance, n_0, n_1 are connected in G_0 and m_0, m_1 are not connected in G_1.

Let n_i, m_i be the tops of the components isomorphic to $[S_i]$, where $i \in \{0, 1\}$. By the construction, if n is the top of an infinite component of G_i, then n is connected to all other tops in G_i. Therefore, $[S_0]$ and $[S_1]$ are finite.

Hence there is a step s_0 such that both $G_0^{s_0}$ and $G_1^{s_0}$ contain the components isomorphic to $[S_0]$, $[S_1]$ with tops n_0, n_1 and m_0, m_1 respectively, and n_0, n_1 are connected in $G_0^{s_0}$. Now, if m_0 and m_1 have not yet been connected, then they will be connected at the next step. This contradiction proves the lemma.

$\square$

Lemma 2.8. *G_0 and G_1 are not computably isomorphic.*

Proof. Let $f : G_0 \to G_1$ be a computable isomorphism. Note that f maps tops to tops, and the component containing the top $2n$ in G_0 is isomorphic to the one containing the top $f(2n)$ in G_1. Therefore, the enumerations $\{A_n^0\}_{n\in\omega}$ and $\{A_n^1\}_{n\in\omega}$ are reducible to one another via the computable functions $h_0(n) = f(2n)/2$ and $h_1(n) = f^{-1}(2n)/2$, which contradicts our choice of $\{A_n^0\}_{n\in\omega}$ and $\{A_n^1\}_{n\in\omega}$.

$\square$

Lemma 2.9. *Let H be a computable graph isomorphic to G, then H is computably isomorphic either to G_0 or to G_1. Thus G has computable dimension two.*

Proof. Since H is computable, there is a computable list $t_0 < t_1 < t_2 < \dots$ of the tops in H, where $<$ is the natural order on ω. The structure H gives rise to a one-to-one computable enumeration $\{A_n\}_{n\in\omega}$ of $\mathcal{F}$ such that $k \in A_n$ iff there is a subgraph of H isomorphic to $[k]$ containing t_n as its top.

Since $\mathcal{F}$ has exactly two nonequivalent one-to-one computable enumerations, $\{A_n\}_{n\in\omega}$ is equivalent either to $\{A_n^0\}_{n\in\omega}$ or $\{A_n^1\}_{n\in\omega}$. Suppose that $\{A_n\}_{n\in\omega}$ is equivalent to $\{A_n^0\}_{n\in\omega}$. We now construct a computable isomorphism h from H to G_0.

By our assumption, there is a computable function f such that $A_n = A^0_{f(n)}$ for all n. Note that f is a permutation of ω because $\{A_n\}_{n\in\omega}$ and $\{A_n^0\}_{n\in\omega}$ are one-to-one. Take any $k \in H$; the value of $h(k)$ is defined according to the following three cases:

1) If $k = t_n$ for some n, then $h(k) = 2f(n)$.

2) If k is the linking element between t_n and t_m, then $h(k)$ is the linking element between $2f(n)$ and $2f(m)$ in G_0. Note that such a linking element exists since $H \cong G_0$.

3) If k is neither a top nor a linking element, then there are m and t_n such that k belongs to the subgraph of H isomorphic to $[m]$ with the top t_n. Let l be the length of the unique path from t_n to k without repetitions. Now, $h(k)$ is the unique element of G_0 belonging to the subgraph isomorphic to $[m]$ with the top $2f(n)$ such that the length of the path from $2f(n)$ to $h(k)$ without repetitions is equal to l.

By the construction, $h : H \to G_0$ is an isomorphism. It is computable since, for a given $k \in H$, one can effectively find out which one of the cases 1), 2) or 3) holds and then effectively find the value of $h(k)$.

$\square$

To show that G is prime we will use the following model-theoretic fact.

Proposition 2.10. *Let $\mathcal{A}$ be a model such that for every $a \in \mathcal{A}$, there is a formula $\varphi_a(x)$ in the language of $\mathcal{A}$ with the property that*

$$\mathcal{A} \models \forall z\ (\varphi_a(z) \leftrightarrow a = z).$$

Then $\mathcal{A}$ is the prime model of its theory.

Lemma 2.11. *G is prime.*

Proof. Due to Proposition 2.10, to prove that G is prime it suffices to show that for every $a \in G$, there is a formula $\varphi_a(x)$ in the language of directed graphs such that $G \models \forall x\ (\varphi_a(x) \leftrightarrow a = x)$. Let $E(x, y)$ be the edge relation on G.

By the construction, the top of every infinite component is connected to all other tops. On the other hand, the top of every finite component is not connected to all other tops. To see this, let $2n_0$ be the top of a finite component $[A^0_{n_0}]$ in G_0, and let n_1 be such that $A^0_{n_0} = A^1_{n_1}$. Hence, $2n_1$ is the top of a finite component isomorphic to $[A^0_{n_0}]$ in G_1. Consider the step s_0 by which we have constructed the components $[A^0_{n_0}]$ and $[A^1_{n_1}]$ in $G^{s_0}_0$ and $G^{s_0}_1$ respectively. Since $\mathcal{F}$ contains infinitely many singletons, there are k_0 and k_1, such that $2k_0$ and $2n_0$ are not connected in $G^{s_0}_0$, $2k_1$ and $2n_1$ are not connected in $G^{s_0}_1$, and $A^0_{k_0}$, $A^1_{k_1}$ are equal one-element sets. Then $2k_0$, $2n_0$ are not connected in G_0 as well as $2k_1$, $2n_1$ are not connected in G_1 because we do not connect $2k_0$ with any top when the only element of $A^0_{k_0}$ is enumerated in it, and the same is true for $2k_1$ in G_1.

First, let us define $\varphi_a(x)$ when a is a top. If a is the top of a finite component $[S]$, then $\varphi_a(x)$ states that $E(x, x)$ and x belongs to a subgraph isomorphic to $[n(S)]$, where $n(S)$ is the marker for the finite set

$S \in \mathcal{F}$. If a is the top of an infinite component $[S]$, then $\varphi_a(x)$ states that $E(x,x)$ & $\forall y$ $(E(y,y) \rightarrow$ "x and y are connected via a linking element"), and x belongs to a subgraph isomorphic to $[n(S)]$, where $n(S)$ is the marker for the infinite set $S \in \mathcal{F}$.

If a is a linking element between two tops u and v, then

$$\varphi_a(x) = \exists y \; \exists z \; (E(x,y) \wedge E(y,x) \wedge E(x,z) \wedge E(z,x) \wedge \varphi_u(y) \wedge \varphi_v(z)).$$

If a is neither a top nor a linking element, then let k, l and u be such that a belongs to the subgraph of G isomorphic to $[k]$ with the top u, and l is the length of the unique path from u to a without repetitions. In this case $\varphi_a(x)$ states that

$$\begin{aligned}\exists z \; (\varphi_u(z) \;\&\; &\text{"}x \text{ belongs to a subgraph isomorphic to } [k] \text{ with top } z\text{"} \;\&\\ &\text{"there is a path of length } l \text{ without repetitions from } z \text{ to } x\text{"}).\end{aligned}$$

□

Theorem 2.1 now follows from Lemmas 2.7, 2.8, 2.9, and 2.11.

3 Codings into another structures

Hirschfeldt, Khoussainov, Shore, and Slinko [7] developed the technique for coding directed graphs into structures like symmetric, irreflexive graphs, partial orders, lattices, rings, 2-step nilpotent groups, and so on. These codings are effective in the sense that they preserve various interesting computability-theoretic properties of the structures such as the computable dimension, the degree spectra of the structures, and the spectra of relations on computable structures.

Our goal in this section is to show that in some cases these codings also preserve the model-theoretic property of being the prime model. For instance, let G be a graph such that every element of it is defined by a first order formula. We will show that the codings of G into a partial order, a lattice, and an integral domain preserve this property. Hence these structures will be the prime models of their theories. However, in the case of integral domains we will need to add finitely many constants.

Let G be the directed graph constructed in the previous section. First, we show how to encode G into a prime symmetric, irreflexive graph H_G of computable dimension two. We then encode H_G into a prime partial order, a lattice, and an almost prime integral domain preserving its computable dimension. The reader can find pictorial diagrams of these codings in [7].

3.1 Symmetric, irreflexive graphs

Let G be an infinite, computable graph, and E be its edge relation. Without loss of generality we will assume that $|G| = \omega$. A computably presentable symmetric, irreflexive graph $H_G = (|H_G|, F)$ is defined as follows.

1. $|H_G| = \{a, a', b, b', b''\} \cup \{c_i, d_i, e_i : i \in \omega\}$.

2. $F(x, y)$ holds only in the following cases.

   (a) $F(a, a')$, $F(a', a)$, $F(b, b')$, $F(b', b)$, $F(b', b'')$, $F(b'', b')$.

   (b) For all $i \in \omega$,

$$\begin{array}{ll} F(a, c_i) \text{ and } F(c_i, a), & F(d_i, e_i) \text{ and } F(e_i, d_i), \\ F(c_i, d_i) \text{ and } F(d_i, c_i), & F(b, e_i) \text{ and } F(e_i, b). \end{array}$$

   (c) If $E(i, j)$ then $F(c_i, e_j)$ and $F(e_j, c_i)$.

Define

$$D(x) = \{x \in |H_G| : x \neq a' \wedge F(a, x)\} = \{c_i : i \in \omega\}$$

and

$$R(x, y) = \{(x, y) : D(x) \wedge D(y) \wedge \exists d, e(F(b, e) \wedge F(e, d) \wedge F(d, y) \wedge F(x, e))\}.$$

Then the mapping $g : i \to c_i$ is an isomorphism from G onto the graph with the domain D^{H_G} and the edge relation $R^{H_G}(x, y)$.

Proposition 3.1. *For any computable presentation of H_G, the sets $D^{H_G} = \{c_i^{H_G} : i \in \omega\}$, $\{d_i^{H_G} : i \in \omega\}$, $\{e_i^{H_G} : i \in \omega\}$, and the relation R^{H_G} are computable.*

Proof. Clearly, $D^{H_G} = \{c_i^{H_G} : i \in \omega\}$, $\{d_i^{H_G} : i \in \omega\}$, and $\{e_i^{H_G} : i \in \omega\}$ are computable since they are definable by quantifier-free formulas with parameters. Hence, R^{H_G} is also computable since for all $x, y \in D^{H_G}$,

$$\exists d, e[F(b, e) \wedge F(e, d) \wedge F(d, y) \wedge F(x, e)] \\ \iff \forall d, e[(F(b, e) \wedge F(e, d) \wedge F(d, y)) \to F(x, e)].$$

□

Proposition 3.2. *The relations D and R are definable by first-order formulas in the language of graphs.*

Proof. It suffices to show that the constants a, a', b, b', and b'' are definable. Let

$$\psi_{b''}(x) = \exists!\, y F(x,y) \wedge \forall y[F(x,y) \rightarrow \exists!\, z(z \neq x \wedge F(z,y))],$$

then $\psi_{b''}$ defines b''. The following formulas define b', b, a', and a respectively:

$$\psi_{b'}(x) = \exists y(F(x,y) \wedge \psi_{b''}(y)), \quad \psi_b(x) = \exists y(F(x,y) \wedge \psi_{b'}(y)) \wedge \neg\psi_{b''}(x),$$
$$\psi_{a'}(x) = \exists!\, y F(x,y) \wedge \neg\psi_{b''}(x), \quad \psi_a(x) = \exists y(F(x,y) \wedge \psi_{a'}(y)).$$

□

Let G be the prime graph of computable dimension two constructed in Section 2, and let G_1, G_2 be its two computable presentations which are not computably isomorphic. For each $j = 1, 2$, let us choose a computable presentation H_{G_j} of H_G such that the isomorphic embedding $g_j : i \rightarrow c_i^{H_{G_j}}$ is computable.

Proposition 3.3. *H_G has computable dimension two.*

Proof. If $f : H_{G_1} \rightarrow H_{G_2}$ is a computable isomorphism, then so is $\widehat{f} = g_2^{-1} \circ f \circ g_1 : G_1 \rightarrow G_2$. Indeed, $E^{G_1}(i,j) \Leftrightarrow R^{H_{G_1}}(g_1(i), g_1(j)) \Leftrightarrow R^{H_{G_2}}(f \circ g_1(i), f \circ g_1(j)) \Leftrightarrow E^{G_2}(g_2^{-1} \circ f \circ g_1(i), g_2^{-1} \circ f \circ g_1(j))$. So, H_{G_1} and H_{G_2} are not computably isomorphic.

Let $H_{G'}$ be any computable presentation of H_G, and let G' be a computable graph with the domain $D^{H_{G'}}$ and the edge relation $R^{H_{G'}}$. Since $H_{G'} \cong H_G$ and D and R are definable relations, we have $G' \cong G$. Hence for some $j = 1, 2$, there is a computable isomorphism $h : G' \rightarrow G_j$. Now we can construct a computable isomorphism φ from $H_{G'}$ to H_{G_j}.

Let $\varphi(a^{H_{G'}}) = a^{H_{G_j}}$, $\varphi(a'^{H_{G'}}) = a'^{H_{G_j}}$, $\varphi(b^{H_{G'}}) = b^{H_{G_j}}$, $\varphi(b'^{H_{G'}}) = b'^{H_{G_j}}$, $\varphi(b''^{H_{G'}}) = b''^{H_{G_j}}$. For every other $x \in |H_{G'}|$, $\varphi(x)$ is defined as follows.

(1) If $x \in D^{H_{G'}}$, that is $x = c_i^{H_{G'}}$ for some $i \in \omega$, let $\varphi(x) = g_j(h(x)) = c_{h(x)}^{H_{G_j}}$.

(2) If $x = d_i^{H_{G'}}$ for some $i \in \omega$, let $\varphi(x) = d_{h(y)}^{H_{G_j}}$, where $y = c_i^{H_{G'}}$ is an element of $D^{H_{G'}}$ which is connected to x. In other words, $\varphi(x)$ is an element of $\{d_i^{H_{G_j}} : i \in \omega\}$ that is connected to $g_j(h(y)) = c_{h(y)}^{H_{G_j}}$.

(3) If $x = e_i^{H_{G'}}$ for some $i \in \omega$, let $\varphi(x) = e_{h(y)}^{H_{G_j}}$, where $y = c_i^{H_{G'}}$.

It is easy to check that this construction for $\varphi : H_{G'} \to H_{G_j}$ is effective. Therefore, H_G has dimension two.

$\square$

Proposition 3.4. *H_G is prime.*

Proof. It suffices to show that every element $x \in |H_G|$ is definable by a first order formula. The formulas that define the constants a, a', b, b', and b'' are given in the proof of Proposition 3.2. Consider c_i; we know that there exists a formula $\varphi_i(x)$ that defines the element $i \in |G|$. Let $\psi_{c_i}(x)$ be the formula obtained from $\varphi_i(x)$ by replacing every occurrence of the binary predicate E with the formula for R, every occurrence of $\forall z \ldots$ with $\forall z(D(z) \to \ldots)$, and every occurrence of $\exists z \ldots$ with $\exists z(D(z) \wedge \ldots)$, where z is any variable. Then $\psi_{c_i}(x)$ defines c_i. Furthermore, d_i is defined by

$$\psi_{d_i}(x) = \neg\psi_a(x) \wedge \exists y(F(x,y) \wedge \psi_{c_i}(y)) \wedge \neg\exists y(F(x,y) \wedge \psi_b(y)),$$

and e_i is defined by $\psi_{e_i}(x) = \exists y(F(x,y) \wedge \psi_{d_i}(y)) \wedge \neg\psi_{c_i}(x)$. Therefore, H_G is prime.

$\square$

3.2 Partial orderings

Let G be an infinite, computable, symmetric, irreflexive graph, and E be its edge relation. Again we assume that $|G| = \omega$. A computably presentable partial ordering $P_G = (|P_G|, \preccurlyeq)$ is defined as follows.

1. $|P_G| = \{a, b\} \cup \{c_i : i \in \omega\} \cup \{d_{i,j} : i < j \in \omega\}$.
2. The relation $\preccurlyeq$ is the smallest partial ordering on $|P_G|$ satisfying the following conditions.
   (a) $a \preccurlyeq c_i \preccurlyeq b$ for all $i \in \omega$.
   (b) If $i < j$ and $E(i,j)$, then $d_{i,j} \preccurlyeq c_i, c_j$.
   (c) If $i < j$ and $\neg E(i,j)$, then $c_i, c_j \preccurlyeq d_{i,j}$.

Define

$$D(x) = \{x \in |P_G| : a \prec x \prec b\} = \{c_i : i \in \omega\}$$

and

$$R(x, y) = \{(x, y) : x \neq y \wedge D(x) \wedge D(y) \wedge \exists z \neq a\,(z \preccurlyeq x, y)\}.$$

Note that $g : i \to c_i$ is an isomorphism from G onto the graph with the domain D^{P_G} and the edge relation $R^{P_G}(x, y)$.

Proposition 3.5. *For any computable presentation of P_G, the relations D^{P_G} and R^{P_G} are computable.*

Proof. Obviously, D^{P_G} is computable, and so is R^{P_G} since for all $x \neq y \in D^{P_G}$,

$$\exists z \neq a\,(z \preccurlyeq x, y) \Longleftrightarrow \neg \exists z \neq b\,(x, y \preccurlyeq z).$$

□

Proposition 3.6. *The relations D and R are definable by first-order formulas in the language of partial orders.*

Proof. It suffices to show that the constants a and b are definable. Let $\psi_a(x) = \forall y\,(\exists z\,(z \prec y) \to x \preccurlyeq y)$ and $\psi_b(x) = \forall y\,(\exists z\,(y \prec z) \to y \preccurlyeq x)$. It is not hard to see that $\psi_a(x)$ and $\psi_b(x)$ define a and b, respectively.

□

Let G be the prime, symmetric, irreflexive graph of computable dimension two constructed in Section 3.1, and let G_1, G_2 be its two computable presentations which are not computably isomorphic. For each $j = 1, 2$, let us choose a computable presentation of P_{G_j} such that the mapping $g_j : i \to c_i^{P_{G_j}}$ is computable.

Proposition 3.7. *P_G has computable dimension two.*

Proof. If $f : P_{G_1} \to P_{G_2}$ is a computable isomorphism, then so is $\widehat{f} = g_2^{-1} \circ f \circ g_1 : G_1 \to G_2$. Therefore, P_{G_1} and P_{G_2} are not computably isomorphic.

Let $P_{G'}$ be any computable presentation of P_G, and let G' be a computable graph with the domain $D^{P_{G'}}$ and the edge relation $R^{P_{G'}}$. Since $P_{G'} \cong P_G$ and D and R are definable relations, we have $G' \cong G$. Hence for some $j = 1, 2$, there is a computable isomorphism $h : G' \to G_j$. A computable isomorphism φ from $P_{G'}$ to P_{G_j} is now defined as follows.

Let $\varphi(a^{P_{G'}}) = a^{P_{G_j}}$ and $\varphi(b^{P_{G'}}) = b^{P_{G_j}}$. For every $x \in D^{P_{G'}}$, that is if $x = c_i^{P_{G'}}$ for some $i \in \omega$, let $\varphi(x) = g_j(h(x)) = c_{h(x)}^{P_{G_j}}$. If $x = d_{i,j}^{P_{G'}}$ for some $i < j$, then either $\exists\, y_1, y_2 \in D^{P_{G'}}(y_1 \neq y_2 \wedge x \preccurlyeq y_1, y_2)$ or $\exists\, y_1, y_2 \in D^{P_{G'}}(y_1 \neq y_2 \wedge y_1, y_2 \preccurlyeq x)$. We can effectively find out which one of the cases holds as well as the relevant y_1, y_2. Suppose $x \preccurlyeq y_1, y_2$; in this case let $\varphi(x)$ be the unique element of P_{G_j} that is less than both $g_j(h(y_1)) = c_{h(y_1)}^{P_{G_j}}$ and $g_j(h(y_2)) = c_{h(y_2)}^{P_{G_j}}$ and that is not equal to $a^{P_{G_j}}$.

It is easy to see that this construction for $\varphi : P_{G'} \to P_{G_j}$ is effective. Therefore, P_G has computable dimension two.

□

Proposition 3.8. *P_G is prime.*

Proof. Let us show that every element of P_G is definable by a first order formula. The formulas that define the constants a and b are given in the proof of Proposition 3.6. Recall that every $i \in |G|$ is defined by some formula $\varphi_i(x)$. Now, every c_i is defined by the formula $\psi_{c_i}(x)$ obtained from $\varphi_i(x)$ by replacing every occurrence of the binary predicate E with the formula for R, every occurrence of $\forall z \ldots$ with $\forall z(D(z) \to \ldots)$, and every occurrence of $\exists z \ldots$ with $\exists z(D(z) \wedge \ldots)$, where z is any variable. If $E(i,j)$, then $d_{i,j}$ is defined by

$$\psi_{d_{i,j}}(x) = \neg\psi_a(x) \wedge \exists\, y_1, y_2(\psi_{c_i}(y_1) \wedge \psi_{c_j}(y_2) \wedge x \preccurlyeq y_1, y_2).$$

If $\neg E(i,j)$, then $d_{i,j}$ is defined by

$$\psi_{d_{i,j}}(x) = \neg\psi_b(x) \wedge \exists\, y_1, y_2(\psi_{c_i}(y_1) \wedge \psi_{c_j}(y_2) \wedge y_1, y_2 \preccurlyeq x).$$

Therefore, P_G is prime.

□

3.3 Lattices

Let G be an infinite, computable, symmetric, irreflexive graph with edge relation E and $|G| = \omega$. A computably presentable lattice $L_G = (|L_G|, \curlywedge, \curlyvee)$ is defined as follows.

1. $|L_G| = \{a, b, k\} \cup \{c_i, m_i : i \in \omega\} \cup \{d_{i,j} : i < j \wedge E(i,j)\}$.

2. For all $x, y \in |L_G|$, if $x \neq y$, then $x \curlyvee y = a$ and $x \curlywedge y = b$, except as required to satisfy the following conditions.

   (a) If $i < j$ and $E(i, j)$, then $c_i \curlyvee c_j = d_{i,j}$.

   (b) $k \curlyvee c_i = m_i$ for all $i \in \omega$.

   (c) $x \curlyvee b = x$ and $x \curlywedge a = x$ for all $x \in |L_G|$.

   Define

$$D(x) = \{x \in |L_G| : (k \curlyvee x \neq a) \wedge (k \curlyvee x \neq x) \wedge x \neq b\} = \{c_i : i \in \omega\}$$

and

$$R(x, y) = \{(x, y) : x \neq y \wedge D(x) \wedge D(y) \wedge (x \curlyvee y \neq a)\}.$$

Note that $g : i \to c_i$ is an isomorphism from G onto the graph with the domain D^{L_G} and the edge relation $R^{L_G}(x, y)$. The following proposition is obvious.

Proposition 3.9. *For any computable presentation of L_G, the relations D^{L_G} and R^{L_G} are computable.*

Let G be the prime, symmetric, irreflexive graph of computable dimension two constructed in Section 3.1. If we add one isolated vertex to G, then the new graph will have the same computable dimension as G, and every element will be definable by a first order formula. This is because G does not have isolated vertices. So, in this section we assume that G has one isolated vertex. Now, let G_1, G_2 be two computable presentations of G that are not computably isomorphic. For each $j = 1, 2$, let us choose a computable presentation of L_{G_j} such that the mapping $g_j : i \to c_i^{L_{G_j}}$ is computable.

Proposition 3.10. *The relations D and R are definable by first-order formulas in the language of lattices.*

Proof. It suffices to show that the constants a, b, and k are definable. The formulas $\psi_a(x) = \forall y(x \curlyvee y = x)$ and $\psi_b(x) = \forall y(x \curlywedge y = x)$ define a and b, respectively. Since G has an isolated vertex, k is the only level-2 element of L_G whose join with any level-2 element is not a. The level-2 elements of L_G are $\{k, c_i : i \in \omega\}$. This can be expressed by the formula

$$\psi_k(x) = \exists z\, (\psi_a(z) \wedge lev_2(x) \wedge \forall y\, (lev_2(y) \to x \curlyvee y \neq z)),$$

where $lev_2(x) = \exists! y\, (x \neq y \wedge x \curlywedge y = y)$.

$\square$

Proposition 3.11. *L_G has computable dimension two.*

Proof. If $f : L_{G_1} \to L_{G_2}$ is a computable isomorphism, then so is $\widehat{f} = g_2^{-1} \circ f \circ g_1 : G_1 \to G_2$. Therefore, L_{G_1} and L_{G_2} are not computably isomorphic.

Let $L_{G'}$ be any computable presentation of L_G, and let G' be a computable graph with the domain $D^{L_{G'}}$ and the edge relation $R^{L_{G'}}$. Since $L_{G'} \cong L_G$ and D and R are definable relations, we have $G' \cong G$. Hence for some $j = 1, 2$, there is a computable isomorphism $h : G' \to G_j$. A computable isomorphism φ from $L_{G'}$ to L_{G_j} is defined as follows.

Let $\varphi(a^{L_{G'}}) = a^{L_{G_j}}$, $\varphi(b^{L_{G'}}) = b^{L_{G_j}}$, and $\varphi(k^{L_{G'}}) = k^{L_{G_j}}$. For every other $x \in |L_{G'}|$, $\varphi(x)$ is defined as follows.

(1) If $x \in D^{L_{G'}}$, that is $x = c_i^{L_{G'}}$ for some $i \in \omega$, let $\varphi(x) = g_j(h(x)) = c_{h(x)}^{L_{G_j}}$.

(2) If $x \notin D^{L_{G'}}$ and $x \curlyvee k^{L_{G'}} \neq a^{L_{G'}}$, that is $x = m_i^{L_{G'}}$ for some $i \in \omega$, then there is $z \in D^{L_{G'}}$ such that $z \curlyvee k^{L_{G'}} = x$. In this case let $\varphi(x) = k^{L_{G_j}} \curlyvee \varphi(z)$.

(3) If $x \notin D^{L_{G'}}$ and $x \curlyvee k^{L_{G'}} = a^{L_{G'}}$, that is $x = d_{i,j}^{L_{G'}}$ for some $i < j$, then there are $z_1, z_2 \in D^{L_{G'}}$ such that $z_1 \curlyvee z_2 = x$. In this case let $\varphi(x) = \varphi(z_1) \curlyvee \varphi(z_2)$.

It is easy to see that this construction for $\varphi : L_{G'} \to L_{G_j}$ is effective. Hence L_G has dimension two.

□

Proposition 3.12. *L_G is prime.*

Proof. We show that every element of L_G is definable by a first order formula. The formulas that define the constants a, b, and k are given in the proof of Proposition 3.10. Let $i \in |G|$ be defined by a formula $\varphi_i(x)$, then c_i is defined by the formula $\psi_{c_i}(x)$ obtained from $\varphi_i(x)$ by replacing every occurrence of the binary predicate E with the formula for R, every occurrence of $\forall z \ldots$ with $\forall z(D(z) \to \ldots)$, and every occurrence of $\exists z \ldots$ with $\exists z(D(z) \wedge \ldots)$, where z is any variable. Each $d_{i,j}$ is defined by

$$\psi_{d_{i,j}}(x) = \exists\, z_1, z_2((x = z_1 \curlyvee z_2) \wedge \psi_{c_i}(z_1) \wedge \psi_{c_j}(z_2)).$$

Each m_i is defined by

$$\psi_{m_i}(x) = \exists\, z_1, z_2((x = z_1 \curlyvee z_2) \wedge \psi_{c_i}(z_1) \wedge \psi_k(z_2)).$$

Therefore, L_G is prime.

$\square$

3.4 Integral domains

Definition 3.13. We say that a model $\mathcal{A}$ is *almost prime* if there is a finite tuple $\bar{a} = a_1, \ldots, a_k$ of elements of $\mathcal{A}$ such that the enriched structure $(\mathcal{A}, \bar{a})$ is the prime model of its own theory.

Let G be a computable symmetric, irreflexive graph with the edge relation E and $|G| = \omega$. Fix a number p which is either 0 or prime. We will use the convention that $\mathbb{Z}_0 = \mathbb{Z}$. Let I be the set of invertible elements of $\mathbb{Z}_p$, which is obviously finite.

The computably presentable integral domain A_G is defined to be

$$\mathbb{Z}_p[x_i : i \in \omega]\Big[\frac{y}{x_i x_j} : E(i,j)\Big]\Big[\frac{z}{x_i x_j} : \neg E(i,j)\Big]\Big[\frac{y}{x_i^n} : i, n \in \omega\Big].$$

From [7] it follows that A_G has the same computable dimension as G if G has the following property: for every finite set of nodes S, there exist nodes $x, y \notin S$ that are connected by an edge. Note that the graph constructed in Section 3.1 satisfies this property. Therefore, for this G, A_G has computable dimension two.

We prove that A_G is almost prime. This will require the following model-theoretic fact, which is a strengthening of Proposition 2.10.

Proposition 3.14. *Let $\mathcal{A}$ be a model in a countable language. Suppose that for every $a \in \mathcal{A}$, there is a formula $\varphi_a(x)$ in the language of $\mathcal{A}$ such that $\mathcal{A} \vDash \varphi_a(a)$ and $\varphi_a(\mathcal{A}) = \{b \in \mathcal{A} : \mathcal{A} \vDash \varphi_a(b)\}$ is finite. Then $\mathcal{A}$ is the prime model of its theory.*

Define

$$D(x) = \{x \in |A_G| : x \notin I \wedge \exists r(x^2 r = z)\},$$
$$Q(x, x') = \{(x, x') : D(x) \wedge \exists a \in I(x' = ax)\},$$

and

$$R(x, x') = \{(x, x') : D(x) \wedge D(x') \wedge \neg Q(x, x') \wedge \exists r(rxx' = y)\}.$$

Let $\varphi_i(x)$ be a formula that defines $i \in |G|$, and let $\psi_i(x)$ be the formula obtained from $\varphi_i(x)$ by replacing every occurrence of the binary predicate

E with the formula for R, every occurrence of the equality relation with the formula for Q, and every occurrence of $\forall z \ldots$ and $\exists z \ldots$ with $\forall z(D(z) \to \ldots)$ and $\exists z(D(z) \wedge \ldots)$, respectively, where z is any variable.

From Lemmas 5.1, 5.2 and Corollary 5.5 of [7] it follows that I can be defined as the set of invertible elements of A_G, $D^{A_G} = \{ax_i : i \in \omega \wedge a \in I\}$, and $R^{A_G} = \{(ax_i, bx_j) : E^G(i,j) \wedge a, b \in I\}$. This means that Q^{A_G} is a congruence relation on (D^{A_G}, R^{A_G}), and the quotient structure of (D^{A_G}, R^{A_G}) modulo Q^{A_G} is isomorphic to (G, E). Therefore, $\psi_i(A_G) = \{ax_i : a \in I\}$. Note that $\psi_i(A_G)$ is finite since so is I.

Let

$$\begin{aligned} Gen =& \{\pm 1\} \cup \{x_i : i \in \omega\} \cup \left\{ \frac{y}{x_i x_j} : E(i,j) \right\} \cup \left\{ \frac{z}{x_i x_j} : \neg E(i,j) \right\} \cup \\ & \left\{ \frac{y}{x_i^n} : i, n \in \omega \right\}. \end{aligned}$$

Every element of A_G can be expressed as a sum of products of elements of *Gen*. Let us add the constants for y and z to the language of rings. Now, for every $g \in Gen$, there is a formula $\psi_g(x)$ in the expanded language such that $A_G \models \psi_g(g)$ and $\psi_g(A_G)$ is finite. The formulas for x_i's are given above. For $y/x_i x_j$ the required formula is $\psi(x) = \exists u_1 \exists u_2\ (\psi_i(u_1) \wedge \psi_j(u_2) \wedge u_1 u_2 x = y)$. It is easy to see that $\psi(A_G)$ is finite. The other cases are similar.

Since every $a \in A_G$ can be expressed as a term involving elements of *Gen*, one can construct a formula $\psi_a(x)$ in the language expanded by new constants for y and z such that $\psi_a(x)$ holds on a in A_G and $\psi_a(A_G)$ is finite. Therefore, due to Proposition 3.14, A_G is almost prime.

References

[1] P. Cholak, S. Goncharov, B. Khoussainov, and R. A. Shore. Computably categorical structures and expansions by constants. *J. Symbolic Logic*, 64(1):13–37, 1999.

[2] S. S. Goncharov. Computable univalent numerations. *Algebra i Logika*, 19(5):507–551, 617, 1980.

[3] S. S. Goncharov. The problem of the number of nonautoequivalent constructivizations. *Algebra i Logika*, 19(6):621–639, 745, 1980.

[4] V. S. Harizanov. The possible Turing degree of the nonzero member in a two element degree spectrum. *Ann. Pure Appl. Logic*, 60(1):1–30, 1993.

[5] D. R. Hirschfeldt. *Degree Spectra of Relations on Computable Structures.* PhD Thesis, Cornell University, 1999.

[6] D. R. Hirschfeldt, B. Khoussainov, and R. A. Shore. A computably categorical structure whose expansion by a constant has infinite computable dimension. *J. Symbolic Logic*, 68(4):1199–1241, 2003.

[7] D. R. Hirschfeldt, B. Khoussainov, R. A. Shore, and A. M. Slinko. Degree spectra and computable dimensions in algebraic structures. *Ann. Pure Appl. Logic*, 115(1-3):71–113, 2002.

[8] B. Khoussainov and R. A. Shore. Computable isomorphisms, degree spectra of relations, and Scott families. Computability theory. *Ann. Pure Appl. Logic*, 93, 98(1-3):153–193, 297–298, 1998, 1999.