Π^0_1-presentations of algebras*

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Abstract

In this paper we study the question as to which computable algebras are isomorphic to non-computable Π^0_1-algebras. We show that many known algebras such as the standard model of arithmetic, term algebras, fields, vector spaces and torsion-free abelian groups have non-computable Π^0_1-presentations. On the other hand, many of this structures fail to have non-computable Σ^0_1-presentation.

1 Introduction

Effectiveness issues in algebra and model theory have been investigated intensively in the last thirty years. One wishes to understand the effective content of model-theoretic and algebraic results, and the interplay between notions of computability, algebra, and model theory. A significant body of work has recently been done in the area, and this is attested by recent series of *Handbooks* and surveys in computable mathematics, computability, and algebra (see, e.g., [1], [4], [3]). An emphasis has been placed on the study of computable models and algebras. These are the structures whose domains are computable sets of natural numbers, and whose atomic diagrams are computable. The study of computable model theory and algebra can naturally be extended to include a wider class of structures. This can be done by postulating that the atomic diagrams or natural fragments of the atomic diagrams are in some complexity class such as Σ^0_n or Π^0_n. These classes of algebras include computably enumerable (c.e.) algebras and co-c.e. algebras which we call Σ^0_1-algebras and Π^0_1-algebras, respectively. Roughly, Σ^0_1-algebras are the ones whose positive atomic diagrams are computably enumerable, and Π^0_1-algebras are the ones whose negative atomic diagrams are computably enumerable. These include finitely presented algebras (e.g. finitely presented groups or rings) and groups generated by finitely many

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computable permutations of ω. There has been some research on Σ^0_1-algebras (see for example [2], [5], [6], [7], [8], [9]) but not much is known about Π^0_1-algebras and their properties. The main goal of this paper is the study of the question as to which computable algebras are isomorphic to non-computable Π^0_1-algebras. Examples we have in mind are typical computable structures such as the arithmetic $(\omega, S, +, \times)$, finitely generated term algebras and fields. We would like to know whether the isomorphism types of these typical computable structures contain non-computable but Π^0_1-algebras. In regard to this, it is worth to note that all these mentioned structures fail to be isomorphic to non-computable Σ^0_1-algebras, and hence existence of non-computable Π^0_1-algebras isomorphic to algebras mentioned is of an independent interest.

Here is a brief outline of the paper. Further in this section, we give the basic definitions of computable, Σ^0_1, and Π^0_1-algebras, and provide some examples. In the second section we provide a theorem that characterizes those Σ^0_1-algebras that are non-computable. As a corollary we obtain that the isomorphism types of finitely generated computable algebras (in particular, the arithmetic and the term algebras) and computable infinite fields fail to have non-computable Σ^0_1-presentations. In the third section we single out a class of algebras and call algebras from that class term-separable. We prove that many known algebras such as the arithmetic, the term algebras, fields and vector spaces are term-separable. Finally, the last section is devoted to showing that all computable term-separable algebras can be made isomorphic to non-computable Π^0_1-algebras.

We now turn to the basic notions of this paper. For the basics of computability theory the reader is referred to Soare [10]. An **algebra** is a structure of a finite purely functional language (signature) σ. Thus, any algebra $\mathcal{A}$ is of the form $(A; f_0^{\mathcal{A}}, \ldots, f_n^{\mathcal{A}})$, where A is a nonempty set called the domain of the algebra, and each $f_i^{\mathcal{A}}$ is a a total operation on the domain A that interprets the function symbol $f_i \in \sigma$. When there is no confusion the operation named by f_i is denoted by the same symbol f_i. We refer to the symbols $f_0, \ldots, f_n$ as the signature of the algebra. Often we call the operations $f_0, \ldots, f_n$ **basic operations** or **functions** (of the algebra $\mathcal{A}$). Presburger arithmetic $(\omega; 0, S, +)$ is an algebra, so are groups, rings, lattices and Boolean algebras. Fundamental structures which arise in computer science such as lists, stacks, queues, trees, and vectors can all be viewed and studied as algebras.

We now define the notion of a term of an algebra $\mathcal{A}$ over a variable set $X = \{x_0, x_1, \ldots\}$.

Definition 1. Let $\mathcal{A} = (A; f_0, \ldots, f_r)$ be an algebra. We define terms of this algebra as formal expressions over a variable set X and domain A as follows. Every element $a \in A$ and variable $x \in X$ is a term. If $t_1, \ldots, t_n$ are terms and $f \in \sigma$ is a function symbol of arity n then $f(t_1, \ldots, t_n)$ is also a term.

As terms are formal expressions formed from the set $A \cup X$ using the signature σ, it makes sense to talk about *syntactic equality* between terms of the algebra $\mathcal{A}$. For instance, examples of terms of the arithmetic $(\omega, S, +, \times)$ are 5, $(x + (7 \times y)) + S(6)$, $2 + 7$ and $7 + 2$. Note that syntactically, the terms $2 + 7$ and $7 + 2$ are distinct. The elements $\bar{a}$ appearing in a term t of the algebra $\mathcal{A}$ are called parameters of t. We write $t(\bar{x}, \bar{a})$ to mean that the variables of term t are among $\bar{x}$ and parameters are among $\bar{a}$.

Consider the set of all terms without parameters. It can be transformed into an algebra in a natural way by declaring the value of f on any tuple $(t_1, \ldots, t_n)$ to be the term $f(t_1, \ldots, t_n)$. This is called the **term algebra** with generator set X.

Let $\mathcal{A} = (A, f_0, \ldots, f_n)$ be an algebra with computable universe. For each term $t = t(\bar{a})$ of $\mathcal{A}$ without free variables introduce a new constant c_t that names the element $t(\bar{a})$. Expand the signature σ by adding to it all these constant symbols. So, elements $a \in A$ may have several constants c naming it. Denote the expanded signature by σ_A. Thus, we have an expansion of $\mathcal{A}$ by constants c.

Definition 2. Consider the expanded algebra $\mathcal{A}$ in the signature σ_A.

1. The **atomic diagram of** $\mathcal{A}$, denoted by $D(\mathcal{A})$ is the set of all expressions of the type $f_i(c_{a_1}, \ldots, c_{a_n}) = f_j(c_{b_1}, \ldots, c_{b_k})$, $f_i(c_{a_1}, \ldots, c_{a_n}) = c_b$, $c_a = c_b$ and their negations which are true in the algebra $\mathcal{A}$. The algebra $\mathcal{A} = (A; f_0, \ldots, f_n)$ is **computable** if its atomic diagram is a computable set.

2. The **positive atomic diagram of** $\mathcal{A}$, denoted by $PD(\mathcal{A})$, is the set of all expressions of the type $f_i(c_{a_1}, \ldots, c_{a_n}) = f_j(c_{b_1}, \ldots, c_{b_k})$, $f_i(c_{a_1}, \ldots, c_{a_n}) = c_b$, and $c_a = c_b$ which are true in the algebra $\mathcal{A}$. The algebra $\mathcal{A} = (A; f_0, \ldots, f_n)$ is **Σ^0_1-algebra** if its positive atomic diagram is a computably enumerable set.

3. The **negative atomic diagram of** $\mathcal{A}$, denoted by $ND(\mathcal{A})$, is the set of all expressions of the type $f_i(c_{a_1}, \ldots, c_{a_n}) \neq f_j(c_{b_1}, \ldots, c_{b_k})$, $f_i(c_{a_1}, \ldots, c_{a_n}) \neq c_b$, and $c_a \neq c_b$ which are true in the algebra $\mathcal{A}$. The algebra $\mathcal{A} = (A; f_0, \ldots, f_n)$ is **Π^0_1-algebra** if its negative atomic diagram is a computably enumerable set.

It is easy to see that the algebra is computable if and only if it is both Σ^0_1 and Π^0_1-algebra. We give now several examples.

Example 1. Let $\mathcal{A} = (A; f_0, \ldots, f_n)$ be an infinite computable algebra. Then it is isomorphic to an algebra $(\omega, h_1, \ldots, h_n)$, where each h_i is a computable function. Clearly all algebras of the form $(\omega, g_1, \ldots, g_n)$, where each g_i is a computable function, are computable.

Example 2. Typical examples of Σ^0_1-algebras are:

(i) The Lindenbaum algebras of computably enumerable first-order theories, such as Peano arithmetic.

(ii) Finitely presented groups, and in fact all finitely presented algebras.

The following two examples provide simple ways of building Π^0_1-algebras.

Example 3. Let $p_1, \ldots, p_n$ be computable permutations of ω. Consider the group G generated by these permutations. Then G is a Π^0_1-algebra. Indeed, if g and g' are elements of this group then their non-equality is confirmed by the existence of an $n \in \omega$ at which $g(n) \neq g'(n)$.

Example 4. Let $\mathcal{A} = (\omega, f_0, \ldots, f_n)$ be a computable algebra. For terms $t(\bar{x})$ and $p(\bar{x})$ we write $t =_{\mathcal{A}} p$ if the values of t and p are equal for all instantiation of variables. Consider the algebra $\mathcal{B}$ obtained by factoring the term algebra with respect to the relation $=_{\mathcal{A}}$. The algebra $\mathcal{B}$ is a Π_1^0-algebra since non-equality between any two terms $t(\bar{x})$ and $p(\bar{x})$ is confirmed by the existence of a tuple $\bar{a} \in A$ at which $t(\bar{a}) \neq p(\bar{a})$.

Example 5. Let $\Sigma = \{0, \ldots, k-1\}$ be a finite alphabet and $L \subseteq \Sigma^*$ be a computable language. Consider a computable algebra $\mathcal{A} = (\Sigma^*, S_0, \ldots, S_{k-1})$, where $S_i(x) = xi$ for every x. Define a congruence relation $\sim_L$ on $\mathcal{A}$ as follows: $x \sim_L y$ iff $\forall u\ (xu \in L \Longleftrightarrow yu \in L)$. Then $\mathcal{A}/\sim_L$ is Π_1^0-algebra.

A Π_1^0-algebra (or Σ_1^0-algebra) $\mathcal{A}$ can be explained as follows. As the negative atomic diagram of $\mathcal{A}$ can be computably enumerated, the set $E = \{(c_a, c_b) \mid c_a = c_b$ is true in the algebra $\mathcal{A}\}$, representing the equality relation in $\mathcal{A}$, is the complement of a c.e. set. Let f be a basic n-ary operation on $\mathcal{A}$. From the definition of a computably enumerable algebra, the operation f can be thought of as a function induced by a computable function, often also denoted by f, which **respects** the E-equivalence classes in the following sense: for all $x_1, \ldots, x_n, y_1, \ldots, y_n$ if $(x_i, y_i) \in E$, then $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in E$. Therefore, a natural way to think about $\mathcal{A}$ is that the elements of $\mathcal{A}$ are E-equivalence classes, and the operations of $\mathcal{A}$ are induced by computable operations. This reasoning suggests another equivalent approach to the definition of Π_1^0-algebra (as well as Σ_1^0-algebra) explained in the next paragraph.

Let E be an equivalence relation on ω. A computable n-ary function f **respects** E if for all natural numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ so that $(x_i, y_i) \in E$, for $i = 1, \ldots, n$, we have $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in E$. Let $\omega(E)$ be the factor set obtained by factorizing ω by E, and let $f_0, \ldots, f_n$ be computable operations on ω which respect the equivalence relation E. An E**-algebra** is then the algebra $(\omega(E), F_0, \ldots, F_n)$, where each F_i is naturally induced by f_i. It is now not hard to show that an algebra $\mathcal{A}$ is a Π_1^0-algebra if and only if $\mathcal{A}$ is an E-algebra for some Π_1^0 equivalence relation E. In a similar way, an algebra $\mathcal{A}$ is a Σ_1-algebra if and only if $\mathcal{A}$ is an E-algebra for some computably enumerable equivalence relation E.

The **isomorphism type** of an algebra $\mathcal{A}$ is the set of all algebras isomorphic to $\mathcal{A}$. We are interested in those algebras whose isomorphism types contain Π_1^0-algebras. We formalize this in the following definitions. An algebra is Π_1^0**-presentable** if it is isomorphic to a Π_1^0-algebra. Note that there is a distinction between Π_1^0-algebras and Π_1^0-presentable algebras. Π_1^0-algebras are given explicitly by Turing machines representing the basic operations and the complement of equality relation of the algebra, while Π_1^0-presentability refers to the property of the isomorphism types of algebras. All these notions make sense for Σ_1^0-presentable algebras as well, and we will use them without explicit definitions.

There are some notational conventions we need to make. Let $\mathcal{A}$ be a Π_1^0-algebra. As the equality relation on $\mathcal{A}$ can be thought of as an equivalence relation (with a c.e. complement) on ω, we can refer to elements of $\mathcal{A}$ as natural numbers keeping in mind that each number n represents the equivalence class (that is, an element of $\mathcal{A}$). Thus, n can be regarded as either an element of $\mathcal{A}$, representing the equivalence class containing n, or the natural number n. The meaning which we use will be clear from the content. Sometimes we denote elements of $\mathcal{A}$ by $[n]$, with $[n]$ representing the equivalence class containing the number n.

2 Failing non-computable Σ_1^0-presentations

This section is for completeness and the main theorem is from [8]. However, we provide more applications of the theorem in order to contrast Σ_1^0 and Π_1^0-presentations of algebras in the last section.

Let $\mathcal{A}$ and $\mathcal{B}$ be Σ_1^0-algebras. A homomorphism h from the algebra $\mathcal{A}$ into the algebra $\mathcal{B}$ is called a **computable homomorphism** if there exists a computable function $f : \omega \to \omega$ such that h is induced by f. In other words, for all $n \in \omega$, we have $h([n]) = [f(n)]$. We call f a representation of h. Clearly, if h is a computable homomorphism then its **kernel**, that is, the set $\{(n, m) \mid h([n]) = h([m])\}$, is computably enumerable. We say that h is **proper** if there are distinct $[n]$ and $[m]$ in $\mathcal{A}$ whose images under h coincide. In this case the image $h(\mathcal{A})$ is called a proper homomorphic image of $\mathcal{A}$.

Our goal is to give a syntactic characterization of Σ_1^0-algebras that are computable. Let $\mathcal{A}$ be a Σ_1^0-algebra. A **fact** is a computably enumerable conjunction $\&_{i\in\omega}\phi_i(\bar{c})$ of sentences, where each $\phi_i(\bar{c})$ is of the form $\forall \bar{x}\psi_i(\bar{x}, \bar{c})$ with $\psi_i(\bar{x}, \bar{c})$ being a negative atomic formula. Call non-computable Σ_1^0-algebras **properly** Σ_1^0. For example, any finitely generated algebra with undecidable equality problem is properly Σ_1^0.

Definition 3. An algebra $\mathcal{A}$ **preserves the fact** $\&_{i\in\omega}\phi_i(\bar{c})$ if $\mathcal{A}$ satisfies the fact and there is a proper homomorphic image of $\mathcal{A}$ in which the fact is true.

The theorem below tells us that properly Σ_1^0-algebras possess many homomorphisms which are well behaved with respect to the facts true in $\mathcal{A}$.

Theorem 1. *A Σ_1^0-algebra $\mathcal{A}$ is properly Σ_1^0 if and only if $\mathcal{A}$ preserves all facts true in $\mathcal{A}$.*

Proof. Assume that $\mathcal{A}$ is a computable algebra. We can make the domain of $\mathcal{A}$ to be ω. Thus, in the algebra $\mathcal{A}$, the fact $\&_{i\neq j}(i \neq j)$ is clearly true. This fact cannot be preserved in any proper homomorphic image of $\mathcal{A}$.

For the other direction, we first note the following. Given elements m and n of the algebra, it is possible to effectively enumerate the minimal congruence relation, denoted by $\eta(m, n)$, of the algebra which contains the pair (m, n). Now note that if $[m] = [n]$ then $\eta(m, n)$ is the equality relation in $\mathcal{A}$. Denote $\mathcal{A}(m, n)$ the factor algebra obtained by factorizing $\mathcal{A}$ by $\eta(m, n)$. Clearly, $\mathcal{A}(m, n)$ is computably enumerable.

Now assume that $\mathcal{A}$ is properly Σ_1^0-algebra and $\&_{i\in\omega}\phi_i(\bar{c})$ is a fact true in $\mathcal{A}$ which cannot be preserved. Hence, for any m and n in the algebra, if $[m] \neq [n]$ then in the factor algebra $\mathcal{A}(m, n)$, the fact $\&_{i\in\omega}\phi_i(\bar{c})$ cannot be satisfied. Therefore, for given m and n, there exists an i such that in the factor algebra $\mathcal{A}(m, n)$ the sentence $\neg\phi_i(\bar{c})$ is true. Now the sentence $\neg\phi_i(\bar{c})$ is equivalent to an existential sentence quantified over a positive atomic formula. Note that existential sentences quantified over positive atomic formulas true in $\mathcal{A}(m, n)$ can be computably enumerated. Hence, in the original algebra $\mathcal{A}$, for all m and n, either $[m] = [n]$ or there exists a an i such that $\neg\phi_i(\bar{c})$ is true in $\mathcal{A}(m, n)$. This shows that the equality relation in $\mathcal{A}$ is computable, contradicting the assumption that $\mathcal{A}$ is a properly Σ_1^0-algebra. The theorem is proved.

There are several interesting corollaries of the theorem above.

Corollary 1. *If $\mathcal{A}$ is properly computably enumerable then any two distinct elements m and n in $\mathcal{A}$ can be homomorphically mapped into distinct elements in a proper homomorphic image of $\mathcal{A}$.*

Indeed, take the fact $m \neq n$ true in $\mathcal{A}$, and apply the theorem.

Call two homomorphic images $h_1(\mathcal{A})$ and $h_2(\mathcal{A})$ of algebra $\mathcal{A}$ **distinct** if congruences induced by h_1 and h_2 are different.

Corollary 2. *If $\mathcal{A}$ is properly computably enumerable then any fact true in $\mathcal{A}$ is true in infinitely many distinct homomorphic images of $\mathcal{A}$. In particular, $\mathcal{A}$ cannot have finitely many congruences.*

Proof. Let ϕ be a fact true in $\mathcal{A}$. By theorem above, there is a homomorphic image $h_1(\mathcal{A})$ in which ϕ is true, and distinct elements m_1 and n_1 in $\mathcal{A}$ for which $h_1(m_1) = h_1(n_1)$. Now consider the fact $\phi \& (m_1 \neq n_1)$, and apply the theorem to this fact. There is a homomorphic image $h_2(\mathcal{A})$ in which $\phi \& (m_1 \neq n_1)$ is true, and distinct elements m_2 and n_2 in $\mathcal{A}$ for which $h_2(m_2) = h(n_2)$. Now consider the fact $\phi \& (m_1 \neq n_1) \& (m_2 \neq n_2)$, and apply the theorem to this fact. The corollary now follows by induction. The corollary is proved.

This theorem can now be applied to provide several algebraic conditions for computable algebras not to have properly Σ^0_1-presentations.

Corollary 3. *In each of the following cases an infinite computably enumerable algebra $\mathcal{A}$ is computable:*

1. *There exists a c.e. sequence (x_i, y_i) such that $[x_i] \neq [y_i]$ for all i, and for any non-trivial congruence relation η there is (x_j, y_j) for which $([x_j], [y_j]) \in \eta$.*

2. *$\mathcal{A}$ has finitely many congruences.*

3. *$\mathcal{A}$ is finitely generated and every non-trivial congruence relation of $\mathcal{A}$ has a finite index.*

4. *No computable field has a properly Σ^0_1-presentation.*

5. *No finitely generated computable algebra has a a properly Σ^0_1-presentation.*

Proof. For Part 1), we see that the fact $\&_{i \in \omega} [x_i] \neq [y_i]$ is true in $\mathcal{A}$. The assumption states that this fact cannot be preserved in all proper homomorphic images of $\mathcal{A}$. Hence $\mathcal{A}$ must be a computable algebra by the theorem above. For part 2), let $\eta_0, \ldots, \eta_k$ be all non-trivial congruences of $\mathcal{A}$; for each η_i take (x_i, y_i) such that $[x_i] \neq [y_i]$ and $([x_i], [y_i]) \in \eta_i$. Then the fact $\&_{i \leq k}([x_i] \neq [y_i])$ is true in $\mathcal{A}$ but cannot be preserved in all proper homomorphic images of $\mathcal{A}$. Thus $\mathcal{A}$ is a computable algebra. For Part 3), consider any two elements $[m]$ and $[n]$ in $\mathcal{A}$ and consider the congruence relation $\eta([m], [n])$ defined in the proof of the theorem. By assumption, $[m] \neq [n]$ iff the algebra $\mathcal{A}(m, n)$ is finite. The set $X = \{(m, n) \mid \mathcal{A}(m, n) \text{ is finite } \}$ is computably enumerable. Hence, the fact $\&_{(m,n) \in X}([m] \neq [n])$ is true in $\mathcal{A}$ but cannot be preserved in any homomorphic image of $\mathcal{A}$. For part 4), consider a computable field $\mathcal{F} = (F; +, \times, 0, 1)$. This algebra has only two congruence relations (both are trivial). Hence by Part 2) $\mathcal{F}$ does not have a proper Σ^0_1-presentation. For the last part assume that $\mathcal{A}$ is computable and a finitely generated algebra. Let $a_1, \ldots, a_n$ be the generators.

Note that for any element $b \in A$ there exists a term t_b over the generating set $\{a_1, \dots, a_n\}$ whose value in $\mathcal{A}$ equals b. Consider the following fact $\&_{b \neq c} t_b \neq t_c$. Clearly this facts is true in the algebra but can't be preserved in all proper homomorphic images of $\mathcal{A}$. Hence all Σ^0_1-presentations of $\mathcal{A}$ fail to be non-computable. The corollary is proved.

Note that from the corollary above finitely generated term algebras, the arithmetic, and infinite computable fields fail to possesses non-computable Σ^0_1-presentations. The last section shows that all these algebras possess non-computable Π^0_1-presentation.

3 Term-Separable algebras

In this section we define term-separable algebras and provide several examples of such algebras.

Definition 4. Let $\mathcal{A} = (A, f_1, \dots, f_r)$ be an algebra. We say that $\mathcal{A}$ is **term-separable** if for every finite set of terms $\{t_1(x, y), \dots, t_n(x, y)\}$ with parameters from A, every $J \subseteq \{1, \dots, n\}^2$ and every $a \in A$ the following holds:

$$\mathcal{A} \models \bigwedge_{\langle k,l \rangle \in J} t_k(a, a) \neq t_l(a, a) \longrightarrow$$
$$\exists b_1 \, \exists b_2 \, (b_1 \neq b_2) \wedge \bigwedge_{\langle k,l \rangle \in J} t_k(b_1, b_2) \neq t_l(b_1, b_2).$$

Proposition 1. *Let $\mathcal{A}$ be an infinite algebra and for every two terms $t_1(x)$ and $t_2(x)$ with parameters from A the set $\{a \in A : \mathcal{A} \models t_1(a) = t_2(a)\}$ is either finite or equals A. Then $\mathcal{A}$ is term-separable.*

Proof. Consider a set of terms $t_1(x, y), \dots, t_n(x, y)$ with parameters from A, and a set $J \subseteq \{0, \dots, n\}^2$ such that

$$\mathcal{A} \models \bigwedge_{\langle k,l \rangle \in J} t_k(a, a) \neq t_l(a, a).$$

Consider the terms $t_1(x, a), \dots, t_n(x, a)$. For each $\langle k, l \rangle \in J$ let $B_{k,l} = \{b \in A : \mathcal{A} \models t_k(b, a) = t_l(b, a)\}$. Since $a \notin B_{k,l}$, $B_{k,l}$ is finite. Then there exists $b \in A \setminus \bigcup_{\langle k,l \rangle \in J} B_{k,l}$ such that $b \neq a$. Hence,

$$\mathcal{A} \models \bigwedge_{\langle k,l \rangle \in J} t_k(b, a) \neq t_l(b, a).$$

The proposition is proved.

In the next proposition we provide several examples of term-separable algebras.

Proposition 2. *The following infinite algebras are term-separable:*

1. *The arithmetic $(\omega, S, +, \times)$.*
2. *The term algebra generated with the generator set X.*

3. Any infinite field.

4. Any torsion-free abelian group.

5. Any infinite vector space over finite field.

Proof. For the arithmetic and infinite computable field, every term $t(x)$ with parameters is equivalent to polynomial with coefficients from the set of natural number or from the field respectively. Every non-zero polynomial has only finitely many zeros. Hence, the condition of proposition 1 holds and this algebras are term-separable. For part 2), consider two terms $t_1(x)$ and $t_2(x)$ such that $\mathcal{A} \models t_1(a) \neq t_2(a)$ for some $a \in A$. Therefore, terms $t_1(a)$ and $t_2(a)$ differ syntactically and, hence, $t_1(x)$ and $t_2(x)$ differ syntactically. So, $\mathcal{A} \models \forall b\ t_1(b) \neq t_2(b)$ and term algebra is term-separable. For part 4), any term $t(x)$ is equal to the expression $nx + a$, where $n \in \mathbb{Z}$ and $a \in A$. Since the group is torsion-free, the equation $t(x) = 0$ has at most one solution if $n \neq 0$ or $a \neq 0$. Proof for the case of vector spaces is similar to above. The proposition is proved.

4 Admitting non-computable Π^0_1-presentations

This section is devoted to the proof of the following result.

Theorem 2. *Let $\mathcal{A} = (A; f_0, \dots, f_r)$ be computable term-separable algebra and* $\mathbf{d}$ *be any c.e. Turing degree. Then $\mathcal{A}$ possesses a Π^0_1-presentation of degree* $\mathbf{d}$*. In particular, it possesses a non-computable Π^0_1-presentation.*

Proof. We will construct required Π^0_1-presentation of $\mathcal{A}$ step-by-step. At the end of step s we have a number n_s and a collection of finite sets $\{C^s_i\}_{i\in\omega}$ such that $C^s_i \neq \varnothing$ for $i \leqslant n_s$, and $C^s_i = \varnothing$ for $i > n_s$. Also we have partial functions $h_1, \dots, h_r$ with $\mathrm{dom}(h_i) \subseteq (\cup_{i\in\omega} C^s_i)^{m_i}$ and $\mathrm{range}(h_i) \subseteq \cup_{i\in\omega} C^s_i$, where m_i is the arity of f_i. Each h_i has the following property: if $\langle c^1_j, c^2_j\rangle \in \eta_s$ for all $j \leqslant m_i$, then $\langle h_i(\bar{c}^1), h_i(\bar{c}^2)\rangle \in \eta_s$, where

$$\forall\, x, y \in \cup_{i\in\omega} C^s_i \quad \langle x, y\rangle \in \eta_s \Longleftrightarrow \exists\, i\ \{x, y\} \subseteq C^s_i.$$

Furthermore, if $t_1(\bar{c}_1)$ and $t_2(\bar{c}_2)$ are terms constructed from the functions $h_1, \dots,\ h_r$ with $\bar{c}_1, \bar{c}_2 \in \cup_{i\in\omega} C^s_i$ that differ syntactically then their values are also different, provided that they are both defined.

Call $g \in \cup_{i\in\omega} C^s_i$ a *ground* element if for every term $t(\bar{x})$, constructed from the functions $h_1, \dots, h_r$, such that $t(\bar{x})$ is not equal to some variable x or constant c, $g \neq t(\bar{c})$ for every tuple $\bar{c} \in \cup_{i\in\omega} C^s_i$. Note that for every $d \in \cup_{i\in\omega} C^s_i$, there exists a unique term $t(\bar{c})$, constructed from the functions $h_1, \dots, h_r$, with a tuple $\bar{c}$ of ground elements, such that $d = t(\bar{c})$. We denote this term by $\widetilde{d}$. Note that if g is a ground element then $\widetilde{g} = g$.

For each $i \leqslant n_s$, we have a triple of ground elements a_i, b_i, e_i that are all distinct. Initially $\{a_i, b_i, e_i\} \subseteq C^s_i$, but in some subsequent step a_i and b_i may move to other sets C^s_j, C^s_k, while e_i is always in C^s_i to ensure that this set will never be empty.

Also the mapping $\psi_s : i \longrightarrow C^s_i$ gives us a partial isomorphism between $\mathcal{A} \cap \{0, \dots, n_s\}$ and $\{C^s_i\}_{i\leqslant n_s}$ in the following sense: for all $i \leqslant r$, for every tuple

$a_1, \ldots, a_{m_i} \in \{0, \ldots, n_s\}$, and for every tuple $c_1, \ldots, c_{m_i}$, such that $c_j \in C^s_{a_j}$, if $h_i(\bar{c})$ is defined then $f_i(\bar{a}) \leqslant n_s$ and $h_i(\bar{c}) \in C^s_{f_i(\bar{a})}$.

Define a function $g_s : \cup C^s_i \longrightarrow \omega$, such that $g_s(a) = i$, if $a \in C^s_i$. Let D be a c.e. set in degree $\mathbf{d}$ and D^s denotes the elements enumerated in D by the step s. When we add a new element during the construction, we always take the least number that has not been used so far.

Step 0. Let $C^0_0 = \{a_0, b_0, e_0\}$ and $n_0 = 0$.

Step s+1. This step has three substeps. At the end of substep l ($l = 1, 2, 3$) we will have constructed the sets $C^{s,l}_i$.

Case A. If for all $i \leqslant n_s$, $i \notin D^s$ or $g_s(a_i) \neq g_s(b_i)$, then

1) Let $n_{s+1} = n_s + 1$ and $C^{s,1}_i = C^s_i$ for $i \leqslant n_{s+1}$.

2) Put new (ground) elements $a_{n_{s+1}}$, $b_{n_{s+1}}$, $e_{n_{s+1}}$ to $C^{s,2}_{n_{s+1}}$ and let $C^{s,2}_i = C^{s,1}_i$ for $i \leqslant n_s$.

3) For every $i \leqslant r$, every tuple $a_1, \ldots, a_{m_i} \in \{0, \ldots, n_{s+1}\}$, such that $f_i(\bar{a}) \leqslant n_{s+1}$, and every tuple $c_1, \ldots, c_{m_i}$, such that $c_j \in C^{s,2}_{a_j}$, if $h_i(\bar{c})$ has not been yet defined then add a new element to $C^{s,3}_{f_i(\bar{a})}$ and declare it to be the value of $h_i(\bar{c})$.

Let $C^{s+1}_i = C^{s,3}_i$ for all $i \leqslant n_{s+1}$.

Case B. If the condition of case A does not hold then take the least i with the property $i \in D^s$ and $g_s(a_i) = g_s(b_i) = i$. Consider the set

$$D = \{t(\bar{c}) : \exists\, d \in \cup C^s_i \text{ such that } \widetilde{d} = t(\bar{c})\}.$$

If $t(\bar{c}) \in D$ then let $t^*(x, y)$ be a term obtained from $t(\bar{c})$ by replacing each occurrence of a_i with x, each occurrence of b_i with y, every parameter c with $g_s(c)$, and every functional symbol h_i with f_i. For example, the terms $t_1 = a_i$ and $t_2 = b_i$ are in D. Then $t^*_1 = x$ and $t^*_2 = y$.

Let $D = \{t_1(\bar{c}_1), \ldots, t_n(\bar{c}_n)\}$ and $J = \{\langle k, l\rangle : \mathcal{A} \vDash \ t^*_k(i,i) \ \neq \ t^*_l(i,i)\}$. By assumption of the theorem there exist $j_1 \neq j_2$ such that

$$\mathcal{A} \vDash \bigwedge_{\langle k,l\rangle \in J} t^*_k(j_1, j_2) \ \neq \ t^*_l(j_1, j_2).$$

Note that we can effectively find the minimal pair of elements with this property because $\mathcal{A}$ is computable. Now,

1) Move every $d = t(\bar{c}) \in \cup C^s_i$ to the set $C^{s,1}_k$, where $k = t^*(j_1, j_2)$. In particular, note that a_i is moved to $C^{s,1}_{j_1}$ and b_i is moved to $C^{s,1}_{j_2}$. Let n_{s+1} be the maximal i such that $C^{s,1}_i \neq \varnothing$.

2) For each $n_s < i \leqslant n_{s+1}$, put new elements a_i, b_i, e_i to $C^{s,2}_i$ and let $C^{s,2}_i = C^{s,1}_i$ for $i \leqslant n_s$.

3) For every $i \leqslant r$, every tuple $a_1, \ldots, a_{m_i} \in \{0, \ldots, n_{s+1}\}$, such that $f_i(\bar{a}) \leqslant n_{s+1}$, and every tuple $c_1, \ldots, c_{m_i}$, such that $c_j \in C^{s,2}_{a_j}$, if $h_i(\bar{c})$ has not been yet defined then add a new element to $C^{s,3}_{f_i(\bar{a})}$ and declare it to be the value of $h_i(\bar{c})$.

Let $C_i^{s+1} = C_i^{s,3}$ for all $i \leqslant n_{s+1}$. This concludes the step $s+1$.

The following lemmas describe some properties of this construction.

Lemma 1. *For all s and every $c, d \in \cup C_i^s$, if $g_s(c) \neq g_s(d)$ then $g_{s+1}(c) \neq g_{s+1}(d)$.*

Proof. Let $\widetilde{c} = t_1(\bar{c}_1)$ and $\widetilde{d} = t_2(\bar{c}_2)$. If we don't split any pair $\{a_i, b_i\}$ at the step $s+1$, then clearly $g_{s+1}(c) = g_s(c) \neq g_s(d) = g_{s+1}(d)$. Suppose that we split $\{a_i, b_i\}$ at the step $s+1$. Consider the terms $t_1^*(x,y)$, $t_2^*(x,y)$. Then $g_{s+1}(c) = t_1^*(j_1, j_2)$ and $g_{s+1}(d) = t_2^*(j_1, j_2)$. Since $t_1^*(i,i) = g_s(t_1(\bar{c}_1)) = g_s(c) \neq g_s(d) = g_s(t_2(\bar{c}_2)) = t_2^*(i,i)$ and we choose $j_1 \neq j_2$ such that they preserve inequality, we have $g_{s+1}(c) \neq g_{s+1}(d)$. The lemma is proved.

Lemma 2. *For all s, $n_s < n_{s+1}$.*

Proof. If we don't split any pair $\{a_i, b_i\}$ at the step $s+1$, then $n_{s+1} = n_s + 1$. Suppose that we split some $\{a_i, b_i\}$ at this step. For each $j \leqslant n_s$, consider a ground element $e_j \in C_j^s$. Also consider ground elements a_i, b_i from C_i^s. By our construction $e_j \in C_j^{s+1}$ for all $j \leqslant n_s$, and $a_i \in C_{j_1}^{s+1}, b_i \in C_{j_2}^{s+1}$. If j_1 or j_2 is less than or equal to n_s, then it equals i. Since $j_1 \neq j_2$, it is impossible that $j_1, j_2 \leqslant n_s$. Hence, $j_1 > n_s$ or $j_2 > n_s$ and, therefore, $n_{s+1} > n_s$. The lemma is proved.

Lemma 3. *For every $i \leqslant r$ and every m_i-tuple $\bar{c}$, there exists a step s at which $h_i(\bar{c})$ is defined. Hence h_i is a total computable function.*

Proof. Take some s_0 such that $\bar{c} \in \cup C_i^{s_0}$. Let $\bar{c} = c_1, \ldots, c_{m_i}$ and consider the terms $\widetilde{c_j} = t_j(\bar{d}_j)$, $j \leqslant m_i$. Take minimal n such that all tuples $\bar{d}_j$, $j \leqslant m_i$, of ground elements belong to the set $\{a_0, b_0, e_0, \ldots, a_n, b_n, e_n\}$. Take $s_1 \geqslant s_0$ such that after step s_1 we do not split any pair $\{a_i, b_i\}$, $i \leqslant n$. This means that for all $s \geqslant s_1$, $g_s(c_j) = g_{s_1}(c_j)$. Let $g_{s_1}(c_j) = a_j$ and take $s_2 \geqslant s_1$ such that $f_i(\bar{a}) \leqslant n_{s_2}$. Such s_2 exists by lemma 2. Now, if $h_i(\bar{c})$ has not been yet defined then, since $c_j \in C_{a_j}^{s_2}$ and $f_i(\bar{a}) \leqslant n_{s_2}$, we will define $h_i(\bar{c})$ at this step. The lemma is proved.

Now, take any $d \in \mathbb{N}$ and consider the term $\widetilde{d} = t(\bar{c})$. There exists a step s_0 after which we do not split any pair $\{a_i, b_i\}$ of ground elements, such that $a_i \in \bar{c}$ or $b_i \in \bar{c}$. Then $g_s(d) = g_{s_0}(d)$ for all $s \geqslant s_0$. This means that there exists a $g(d) = \lim_s g_s(d)$. Let $C_i = \{d : g(d) = i\}$. Note that $C_i \neq \varnothing$ because $e_i \in C_i$.

Lemma 4. *At each step s the following properties hold:*

(i) *for every $i \leqslant r$ and every m_i-tuples $\bar{c}^1$ and $\bar{c}^2$, such that $g_s(\bar{c}^1) = g_s(\bar{c}^2)$, if $h_i(\bar{c}^1)$ and $h_i(\bar{c}^2)$ are both defined then $g_s(h_i(\bar{c}^1)) = g_s(h_i(\bar{c}^2))$,*

(ii) *$\psi_s : i \longrightarrow C_i^s$ is a partial isomorphism between $\mathcal{A} \cap \{0, \ldots, n_s\}$ and $\{C_i^s\}_{i \leqslant n_s}$.*

Proof. First, note that (ii) implies (i). Now, prove (ii) by induction on s. It suffices to prove the following statement:

> for every $i \leqslant r$, every m_i-tuple $\bar{a}$ and every m_i-tuple $\bar{c}$, such that $c_j \in C_{a_j}^{s,1}$, if $h_i(\bar{c})$ is defined then $f_i(\bar{a}) \leqslant n_{s+1}$ and $h_i(\bar{c}) \in C_{f_i(\bar{a})}^{s,1}$.

This is because, when we put new elements to $C_i^{s,2}$ or $C_i^{s,3}$, we do it according to partial isomorphism.

If we do not split any pair of ground elements at the step $s+1$, then there is nothing to prove. Suppose that we split $\{a_i, b_i\}$ at this step. Then we move every d such that $\widetilde{d} = t(\bar{c})$ to the set $C_k^{s,1}$, where $k = t^*(j_1, j_2)$.

Take any m_i-tuple $\bar{c}$ such that $c_j \in C_{a_j}^{s,1}$ and $h_i(\bar{c})$ is defined. Let $\widetilde{c_j} = t_j(\bar{u}_j)$. Then by construction $a_j = t_j^*(j_1, j_2)$. So, we have

$$g_{s+1}(h_i(\bar{c})) = g_{s+1}(h_i(t_1(\bar{u}_1), \ldots, t_{m_i}(\bar{u}_{m_i}))) = f_i(t_1^*(j_1, j_2), \ldots, t_{m_i}^*(j_1, j_2)) = f_i(\bar{a}).$$

Also note that $f_i(\bar{a}) \leqslant n_{s+1}$ by the choice of n_{s+1}. The lemma is proved.

Consider a relation η defined as follows:

$$\langle x, y\rangle \in \eta \Longleftrightarrow g(x) = g(y).$$

Lemma 5. *η is a congruence relation on $(\mathbb{N}, h_1, \ldots, h_r)$ and $(\mathbb{N}, h_1, \ldots, h_r)/\eta$ is isomorphic to $\mathcal{A}$.*

Proof. Obviously, η is an equivalence relation. Now, take any h_i and two m_i-tuples $\bar{c}^1$ and $\bar{c}^2$ such that $g(\bar{c}^1) = g(\bar{c}^2)$. Take s_0 such that $h_i(\bar{c}^1)$ and $h_i(\bar{c}^2)$ are defined at step s_0 and

$$\forall s \geqslant s_0 \quad g_s(\bar{c}^1) = g(\bar{c}^1) \ \& \ g_s(\bar{c}^2) = g(\bar{c}^2) \ \& \ g_s(h_i(\bar{c}^1)) = g(h_i(\bar{c}^1)) \ \& \ g_s(h_i(\bar{c}^2)) = g(h_i(\bar{c}^2)).$$

From lemma 4(i) it follows that $\forall s \geqslant s_0 \quad g_s(h_i(\bar{c}^1)) = g_s(h_i(\bar{c}^2))$ and, hence, $g(h_i(\bar{c}^1)) = g(h_i(\bar{c}^2))$. So, η is a congruence.

Recall that $C_i = \{d : g(d) = i\}$. Now, prove that the mapping $\psi : i \longrightarrow C_i$ gives us an isomorphism between $\mathcal{A}$ and $(\mathbb{N}, h_1, \ldots, h_r)/\eta$. Take any m_i-tuple $\bar{a}$ and m_i-tuple $\bar{c}$ such that $g(\bar{c}) = \bar{a}$. We need to prove that $g(h_i(\bar{c})) = f_i(\bar{a})$.

Take s_0 such that $h_i(\bar{c})$ is defined at step s_0 and

$$\forall s \geqslant s_0 \quad g_s(\bar{c}) = g(\bar{c}) \text{ and } g_s(h_i(\bar{c})) = g(h_i(\bar{c})).$$

From lemma 4(ii) it follows that $g_s(h_i(\bar{c})) = f_i(\bar{a})$ for all $s \geqslant s_0$. Hence, $g(h_i(\bar{c})) = f_i(\bar{a})$. The lemma is proved.

Lemma 6. *η is Π_1^0 relation whose Turing degree is* $\mathbf{d}$.

Proof. Show that $\mathbb{N}^2 \setminus \eta$ is Σ_1^0. We have

$$\langle x, y\rangle \notin \eta \Longleftrightarrow g(x) \neq g(y) \Longleftrightarrow \exists s\,(x, y \in \cup C_i^s \ \& \ g_s(x) \neq g_s(y)),$$

where the second equivalence follows from lemma 1. Hence, η is Π_1^0.

Now, prove that the degree of η is $\mathbf{d}$. From the construction of theorem 2 it follows that $i \in D$ iff $\langle a_i, b_i\rangle \notin \eta$ and, therefore, $D \leqslant_T \eta$. Show that $\eta \leqslant_T D$. Take any two numbers x, y and find the least s such that $x, y \in \cup_{i \leqslant n_s} C_i^s$. Let $\widetilde{x} = t_1(\bar{d}_1)$ and $\widetilde{y} = t_2(\bar{d}_2)$, where $\bar{d}_1, \bar{d}_2 \in \{a_0, b_0, c_0, \ldots, a_{n_s}, b_{n_s}, c_{n_s}\}$. Find the least $s_1 \geqslant s$ such that we have split all pairs $\{a_i, b_i\}$ for $i \in D \cap \{0, \ldots, n_s\}$ by the step s_1. Then $\langle x, y\rangle \in \eta$ iff $g_{s_1}(x) = g_{s_1}(y)$. The lemma is proved.

Thus, we have proved the theorem.

This theorem and the proposition 2 together give us the following examples of computable algebras that admit non-computable Π_1^0-presentations.

Corollary 4. *The following algebras possess non-computable* Π^0_1*-presentations:*

1. *The arithmetic* $(\omega, S, +, \times)$.
2. *The term algebra generated with the generator set* X.
3. *Any infinite computable field* $(F, +, \times, 0, 1)$.
4. *Any computable torsion-free abelian group.*
5. *Any infinite computable vector space over finite field.*

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