# Membership problem for $2 \times 2$ integer matrices

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joint work with Igor Potapov

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24 July, 2017

This work was supported by EPSRC grant "Reachability problems for words, matrices and maps" (EP/M00077X/1)

# Definitions

## Membership problem

Let $M$ be an $n \times n$ matrix and $F = \{M_1, \ldots, M_k\}$ be a finite collection of $n \times n$ matrices. Determine whether $M \in \langle F \rangle$, that is, whether

$$M = M_{i_1} M_{i_2} \cdots M_{i_t}$$

for some sequence of matrices $M_{i_1}, M_{i_2}, \ldots, M_{i_t} \in F$.

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- It is a long standing open question whether the Membership problem is decidable for $2 \times 2$ matrices (even over integers).

# Main result

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Given a finite collection $F$ of nonsingular matrices from $\mathbb{Z}^{2\times 2}$ and a nonsingular matrix $M \in \mathbb{Z}^{2\times 2}$, it is decidable whether $M \in \langle F \rangle$.

Let $F = \{M_1, \ldots, M_k\} \cup \{N_1, \ldots, N_r\}$,
where $\det(M_i) \neq \pm 1$ and $N_i \in \mathrm{GL}(2, \mathbb{Z})$.

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$M \in \langle F \rangle$ iff there exist $i_1, \ldots, i_t \in \{1, \ldots, k\}$ and matrices $A_1, \ldots, A_t, A_{t+1} \in \mathcal{S}$ such that

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$$M = A_1 M_{i_1} A_2 M_{i_2} \cdots A_t M_{i_t} A_{t+1}.$$

The value of $t$ is bounded: since $|\det(M_i)| \geq 2$, we have that $t \leq \log_2 |\det(M)|$.

So, to decide whether $M \in \langle F \rangle$ we go through all sequences $i_1, \ldots, i_t \in \{1, \ldots, k\}$ of length up to $\log_2 |\det(M)|$ and for each such sequence check whether there are matrices $A_1, \ldots, A_t, A_{t+1} \in \mathcal{S}$ such that

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$$M = A_1 M_{i_1} A_2 M_{i_2} \cdots A_t M_{i_t} A_{t+1}.$$

**Theorem**

*Given a nonsingular matrices $M$ and $M_1, \ldots, M_t$ and a finitely generated semigroup $\mathcal{S} \subseteq \mathrm{GL}(2, \mathbb{Z})$, it is decidable whether there are matrices $A_1, \ldots, A_t, A_{t+1} \in \mathcal{S}$ such that*

$$M = A_1 M_1 A_2 M_2 \cdots A_t M_t A_{t+1}.$$

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**Theorem (Smith Normal Form)**

*For any matrix $A \in \mathbb{Z}^{2\times 2}$, there are matrices $E, F$ from $\mathrm{GL}(2,\mathbb{Z})$ such that $A = E \begin{bmatrix} m & 0 \\ 0 & nm \end{bmatrix} F$ for some $n, m \in \mathbb{N}$.*

*The numbers $n$ and $m$ are uniquely defined by $A$. The diagonal matrix $D = \begin{bmatrix} m & 0 \\ 0 & nm \end{bmatrix}$ is called the Smith normal form of $A$.*

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If $M = A_1 M_1 A_2$, then $M$ and $M_1$ must have the same Smith normal form $D$ because $A_1, A_2 \in \mathrm{GL}(2, \mathbb{Z})$.

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**Theorem**

*Given a matrix* $D = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}$ *and a finitely generated semigroup* $\mathcal{S} \subseteq \mathrm{GL}(2, \mathbb{Z})$*, it is decidable whether there are matrices* $A_1, A_2 \in \mathcal{S}$ *such that*

$$D = A_1 D A_2.$$

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$D = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}$. The equation $D = A_1 D A_2$ is equivalent to

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Let $A_1 = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $A_2 = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, then we have

$$\begin{bmatrix} b_4 & -b_2 \\ -b_3 & b_1 \end{bmatrix} = \begin{bmatrix} a_1 & n a_2 \\ \frac{1}{n} a_3 & a_4 \end{bmatrix}$$

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Let $H = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z}) \ : \ n \text{ divides } a_3 \right\}$.

Thus $A \in H$ if and only if $A^D \in \mathrm{GL}(2, \mathbb{Z})$.

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The index is $H$ in $\mathrm{GL}(2,\mathbb{Z})$ is the number of right cosets of $H$.

The group $\mathrm{GL}(2, \mathbb{Z})$ is generated by the matrices $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

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This presentation is not unique because $S^2 = R^3 = -I$.

However, for every $M \in \mathrm{GL}(2,\mathbb{Z})$ there is a unique canonical word $w \in \{S, R, N\}^*$ which represents $M$.

A word $w$ is canonical if it does not contain subwords $SS$ and $RRR$ and $N$ appears only in the first position of $w$.

# Regular subsets

A subset $\mathcal{S}$ of $\mathrm{GL}(2,\mathbb{Z})$ is regular if there is a regular language $L$ in the alphabet $\Sigma = \{S, R, N\}$ such that

- Every word $w \in L$ represents a matrix from the subset $\mathcal{S}$.
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A semigroup $\mathcal{S} = \langle M_1, \ldots, M_k \rangle$ is defined by the regular expression $(w_1 + \cdots + w_k)^*$, where $w_1, \ldots, w_k$ are words that represent the matrices $M_1, \ldots, M_k$.

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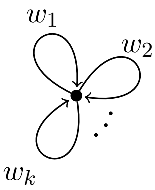

**Theorem**

*The set $L = \{w \;:\; w$ represents a matrix from $H\}$ is regular.*

# $H = \{A \in \mathrm{GL}(2,\mathbb{Z}) \ : \ A^D \in \mathrm{GL}(2,\mathbb{Z})\}$

**Theorem**

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- Let $U_0 = I, U_1, \ldots, U_s$ be representatives of the right cosets of $H$ in $\mathrm{GL}(2,\mathbb{Z})$.

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- Let $U_0 = I, U_1, \ldots, U_s$ be representatives of the right cosets of $H$ in $\mathrm{GL}(2,\mathbb{Z})$.
- Then the automaton $\mathcal{A}$ that recognizes $L$ has the states $Q = \{U_0, U_1, \ldots, U_s\}$, where $U_0$ is both the initial and the final state of $\mathcal{A}$.

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- $\mathcal{A}$ has a transition $U_i \xrightarrow{R} U_j$ iff $U_i R U_j^{-1} \in H$.
  And similarly for $S$- and $N$-transitions.

# Example of an automaton for $H$

Let $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then $H$ has index 3 in $\mathrm{GL}(2, \mathbb{Z})$ and representatives of the right cosets of $H$ in $\mathrm{GL}(2, \mathbb{Z})$ are

$$U_0 = I, \quad U_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } U_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

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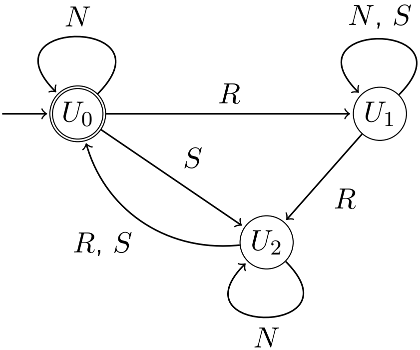


$$U_0 R U_1^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \in H.$$

So, $\mathcal{A}$ has a transition $U_0 \xrightarrow{R} U_1$.

# Base case $D = A_1 D A_2$

Let $\mathcal{M}$ be an automaton that recognizes the semigroup $\mathcal{S}$. We will construct an automaton $\mathrm{Inv}(\mathcal{M})$ that recognizes $\mathcal{S}^{-1}$ and an automaton $\mathcal{F}(\mathcal{M})$ that recognizes the following subset of $\mathrm{GL}(2, \mathbb{Z})$

$$\mathcal{S}^D = \{A^D \ : \ A \in \mathcal{S} \text{ and } A \in H\},$$

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Then the equation $A_2^{-1} = A_1^D$ has a solution $A_1, A_2 \in \mathcal{S}$ iff

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Then the equation $A_2^{-1} = A_1^D$ has a solution $A_1, A_2 \in \mathcal{S}$ iff

$$L(\mathrm{Can}(\mathrm{Inv}(\mathcal{M}))) \cap L(\mathrm{Can}(\mathcal{F}(\mathcal{M}))) \neq \emptyset.$$

$\mathrm{Can}(\mathcal{M})$ recognizes the same subset of $\mathrm{GL}(2,\mathbb{Z})$ as $\mathcal{M}$ but accepts only canonical words.

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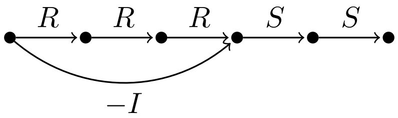

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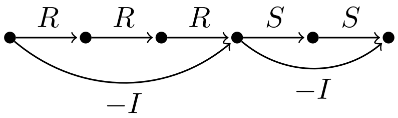

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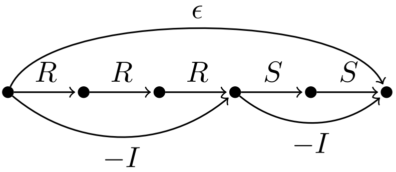

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However, $S^D$ and $R^D$ have fractional coefficients. So they don't belong to $\mathrm{GL}(2,\mathbb{Z})$ and cannot be presented by words.

$$\text{If } D = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}, \text{ then } \ S^D = \begin{bmatrix} 0 & -1 \\ \frac{1}{n} & 0 \end{bmatrix} \ \text{ and } \ R^D = \begin{bmatrix} 0 & -1 \\ \frac{1}{n} & 1 \end{bmatrix}$$

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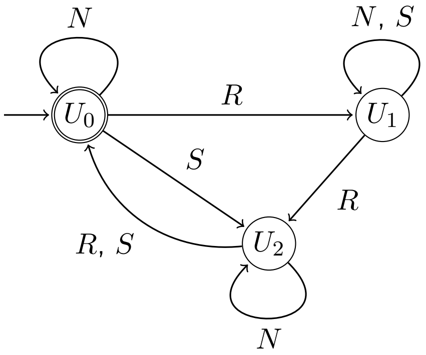
Hence $(U_j R U_m^{-1})^D$ belongs to $\mathrm{GL}(2, \mathbb{Z})$, and we can find a word $w$ that represents $(U_j R U_m^{-1})^D$.

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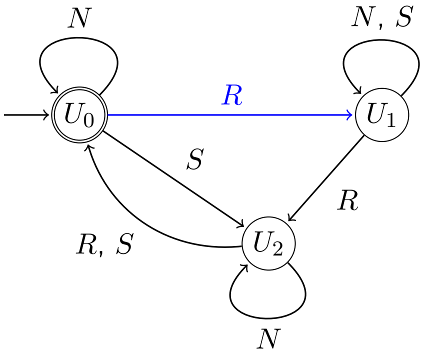


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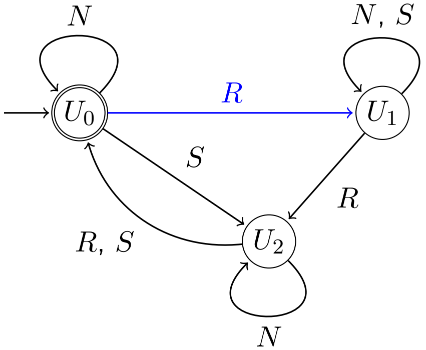


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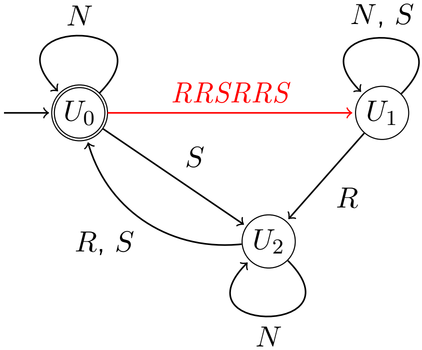


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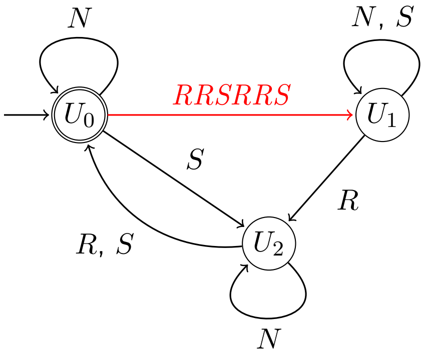


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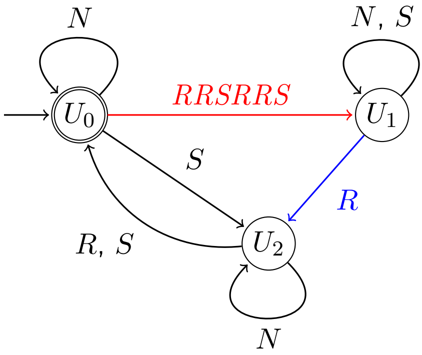


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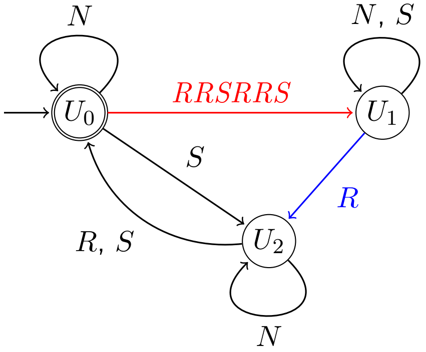


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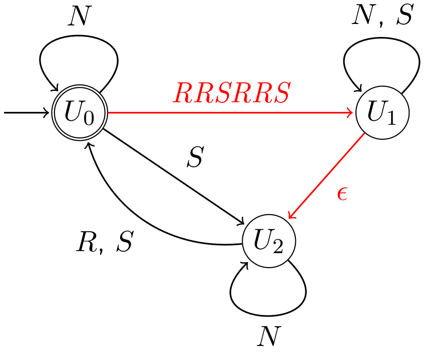


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- The equation $A_2^{-1} = A_1^D$ has a solution $A_1, A_2 \in \mathcal{S}$ iff

$$L(\mathrm{Can}(\mathrm{Inv}(\mathcal{M}))) \cap L(\mathrm{Can}(\mathcal{F}(\mathcal{M}))) \neq \emptyset.$$

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- Express $(SRS)^D = (U_0 S U_1^{-1})^D (U_1 R U_2^{-1})^D (U_2 S U_0^{-1})^D$
- Replace transitions of $\mathcal{M} \times \mathcal{A}$ with new paths:

# Construction of $\mathcal{F}(\mathcal{M})$

Recall that $\mathcal{M}$ recognizes $\mathcal{S}$, and $\mathcal{A}$ recognizes $H$.
To construct $\mathcal{F}(\mathcal{M})$:

- First, construct the product $\mathcal{M} \times \mathcal{A}$ for $L(\mathcal{M}) \cap L(\mathcal{A})$.
- Use $\mathcal{A}$ to rewrite any word accepted by $\mathcal{A}$ as

$$SRS = (U_0 S U_1^{-1})(U_1 R U_2^{-1})(U_2 S U_0^{-1})$$

  where $U_0 \xrightarrow{S} U_1 \xrightarrow{R} U_2 \xrightarrow{S} U_0$ is an accepting run of $\mathcal{A}$.
- Express $(SRS)^D = (U_0 S U_1^{-1})^D (U_1 R U_2^{-1})^D (U_2 S U_0^{-1})^D$
- Replace transitions of $\mathcal{M} \times \mathcal{A}$ with new paths: every transition $(q_i, U_j) \xrightarrow{R} (q_l, U_m)$ is replaced by a path with label $w$, where $w$ represents the matrix $(U_j R U_m^{-1})^D$.