The degree spectra of definable relations on Boolean algebras *

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Abstract

In this paper, we study some questions concerning the structure of the degree spectra of the sets of atoms and atomless elements in a computable Boolean algebra. We prove that if the degree spectrum of the set of atoms contains a 1-low Turing degree, then it contains the computable degree. We also show that in a computable Boolean algebra of characteristic $(1, 1, 0)$ with computable set of atoms the spectrum of the atomless ideal consists of all Π^0_2 Turing degrees.

§1. Introduction. The study of Turing degree spectra of relations on computable models is one of the main topics in computable model theory. It began with the paper [1] by Ash and Nerode which gave a syntactic characterization of intrinsically computable and intrinsically computably enumerable relations.

The research on degree spectra of relations not only provided useful methods for distinguishing different computable presentations of a given model but also grew into an independent and quite a fruitful area connected with various branches of computability theory and mathematical logic.

Following Harizanov's dissertation [6], we will call the *degree spectrum*, or simply the *spectrum*, of a relation R on a computable model A the set

$$\mathrm{Spec}(R) = \{\deg(R') : R' \text{ is the image of } R \text{ in some computable model } A' \cong A\},$$

where $\deg(R')$ is the Turing degree of R'.

Of special interest are the spectra of relations on linear orders and Boolean algebras, since they are sufficiently nontrivial and well studied classes of models.

Recently Downey, Goncharov, and Hirschfeld have completely resolved the question of the cardinality of the degree spectra of computable relations on Boolean algebras.

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Proposition 1 ([4])**.** *Let R be a computable relation on a computable Boolean algebra B. Then either R is definable by a quantifier-free formula with parameters in B (in which case R is intrinsically computable) or* $\mathrm{Spec}(R)$ *is infinite.*

There is a similar result for linear orders:

Proposition 2 (Hirschfeldt)**.** *Let R be a computable relation on a computable linear order L. Then either R is intrinsically computable or* $\mathrm{Spec}(R)$ *is infinite.*

An important role plays the study of the degree spectra of first-order definable relations such as the set of adjacent elements in linear orders and the sets of atoms and atomless elements in Boolean algebras. Remmel [8] proved that the spectrum of the set of atoms in a computable Boolean algebra is closed upwards, provided that it is non-trivial; Downey [2] showed that every such spectrum always contains an incomplete Turing degree. This work continues the study of the degree spectra of the sets of atoms and atomless elements in computable Boolean algebras.

The main definitions from recursion theory, the theory of computable models and Boolean algebras can be found in the books by Rogers [9], Soare [10], Ershov and Goncharov [5], and Goncharov [3].

A set $A \leqslant_T \varnothing'$ is called *1-low*, if $A' \equiv_T \varnothing'$, where A' is the Turing jump of A. The quantifier $\forall^{\#}$ means "for all but finitely many."

When working with binary trees, we will use notations from Goncharov [3]. Define the following functions and relations on $\mathbb{N}$:

$$R(n) = 2n+2, \quad L(n) = 2n+1,$$
$$H(0) = 0 \text{ and } H(n) = [(n-1)/2] \text{ if } n > 0, \text{ where } [x] \text{ is the integer part of } x,$$

$$S(n) = \begin{cases} n-1, & n \text{ is even, } n > 0, \\ n+1, & n \text{ is odd, } n > 0, \\ 0, & n = 0, \end{cases}$$

$$h(0) = 0, \quad h(n+1) = h(H(n+1)) + 1,$$
$$H(x,0) = x, \quad H(x,n+1) = H(H(x,n)),$$
$$x \preccurlyeq y \iff \prod_{n=0}^{h(x)} |H(x,n) - y| = 0.$$

For every $n \in \mathbb{N}$, they have the following meaning:

$R(n)$ is the right child of n,

$L(n)$ is the left child of n,

$H(n)$ is the parent vertex of n if $n \neq 0$,

$S(n)$ is the neighbour of n below $H(n)$,

$h(x)$ is the distance between 0 and n,

$x \preccurlyeq y$ iff x and y are on the same branch and x is below y.

A subset $D \subseteq \mathbb{N}$ is called a *tree* if for every n in D, $H(n)$ and $S(n)$ are in D.

Given a Boolean algebra B and an $x \in B$, denote by $At_B(x)$ the number of atoms of B below x. Also denote the ideal generated by the Fréchet ideal and the atomless ideal by $S(A)$.

To prove that two Boolean algebras are isomorphic, we will often use Vaught's Criterion, which can be found in [3].

In Section 2 we will prove that for Boolean algebras of a certain type, the spectrum of the atomless ideal is complete, i.e., it contains all Π^0_2 Turing degrees. In Section 3 we will prove that if the spectrum of the set of atoms contains a 1-low Turing degree, then it contains the computable degree. In particular, this implies that there is no computable Boolean algebra with the spectrum of the set of atoms consisting of all non-computable c.e. Turing degrees.

§2. Degree spectra of atomless elements. The main result of this section is the following theorem.

Theorem 3. *Let B be a computable Boolean algebra of elementary characteristic $(1,1,0)$ with computable set of atoms. Then for every Π^0_2 set C, there is a computable Boolean algebra $B' \cong B$ such that $Al(B') \equiv_T C$.*

Proof. Goncharov and Vlasov [11] proved that each computable Boolean algebra of elementary characteristic $(1,1,0)$ with computable set of atoms has a decidable presentation. Therefore, we assume that B is decidable. We will need the following definition.

Definition 4. Let $\{B_i\}_{i\in\omega}$ be a sequence of Boolean algebras such that

1) the set $\{(i,x) : x \in B_i\}$ is computably enumerable,
2) the functions $f_0(i,x,y) = x \vee_i y$, $f_1(i,x,y) = x \wedge_i y$, $f_2(i,x) = C_i(x)$ are partially computable, where $\vee_i$, $\wedge_i$, C_i are the basic operations on B_i,
3) the functions $g_0(i) = \mathbf{0}^{B_i}$ and $g_1(i) = \mathbf{1}^{B_i}$ are computable.

We will call such a sequence *computable.*

Consider the set $A = \{\langle x_0,\dots,x_k\rangle : x_i \in B_i,\ x_k = \mathbf{0}^{B_k}$ or $x_k = \mathbf{1}^{B_k}$, and $x_k \neq x_{k-1}\}$. Each tuple $\langle x_0,\dots,x_k\rangle \in A$ defines the unique element $x \in \sum_{i\in\omega}^{\{\mathbf{0},\mathbf{1}\}} B_i$ such that

$$x(i) = \begin{cases} x_i & \text{if } i \leqslant k,\\ \mathbf{0}^{B_i} & \text{if } i > k \text{ and } x_k = \mathbf{0}^{B_k},\\ \mathbf{1}^{B_i} & \text{if } i > k \text{ and } x_k = \mathbf{1}^{B_k}.\end{cases}$$

It is clear that A is computably enumerable. So there exists a one-to-one computable function f such that $\rho f = A$. Using f, define computable functions $\vee$, $\wedge$ and C such that $B = (\mathbb{N};\ \vee,\wedge,C)$ is a computable Boolean algebra isomorphic to $\sum_{i\in\omega}^{\{\mathbf{0},\mathbf{1}\}} B_i$. We call such B a *natural computable presentation* of $\sum_{i\in\omega}^{\{\mathbf{0},\mathbf{1}\}} B_i$.

Let us prove an auxiliary lemma.

Lemma 5. *Let A and B be countable Boolean algebras such that:*

(a) *the set of atoms in both A and B is infinite,*

(b) *neither A nor B contains an infinite atomic element,*

(c) *for every $x \in A$ ($x \in B$) either $x \in S(A)$ ($x \in S(B)$) or $C(x) \in S(A)$ ($C(x) \in S(B)$).*

Then $A \cong B \cong B_{\omega+\eta}$.

Proof. Define

$$S = \{(x, y) : \ x \in A, \ y \in B, \ x \in S(A) \iff y \in S(B), \\ x \in Fr(A) \iff y \in Fr(B) \text{ and } x \in S(A) \Longrightarrow At_A(x) = At_B(y)\}.$$

It is easy to check that S is a condition of isomorphism for A and B. Therefore, by Vaught's Criterion A and B are isomorphic. The algebra $B_{\omega+\eta}$ obviously satisfies the conditions (a)–(c) of Lemma 5. Hence, both A and B are isomorphic to $B_{\omega+\eta}$. □

Lemma 6. *Let B be a decidable Boolean algebra of elementary characteristic $(1, 1, 0)$. Then there exists a computable sequence $\{B_i\}_{i\in\omega}$ of computable Boolean algebras such that $B \cong \sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} B_i$, $\mathrm{ch}_1(B_i) = 0$ and the sets of atoms and atomless elements in B_i are uniformly computable in i.*

Proof. Since B is decidable, the Ershov-Tarski ideal $I(B)$ is computable. Let $\{a_i\}_{i\in\omega}$ be a computable sequence that enumerates all the elements of B. Construct a computable sequence $\{b_i\}_{i\in\omega}$ as follows: let $b_0 = a_t$ for the minimal t such that $a_t \in I(B)$ and $a_t \neq \mathbf{0}$. Suppose $b_0, \ldots, b_n$ have been already constructed; let $b_{n+1} = a_t \setminus \bigvee_{i\leqslant n} b_i$ for the minimal t such that $a_t \in I(B)$ and $a_t \setminus \bigvee_{i\leqslant n} b_i \neq \mathbf{0}$. Let $B_i = \widehat{b_i}$; then $\{B_i\}_{i\in\omega}$ is the required sequence. □

Consider the sequence $\{B_i\}_{i\in\omega}$ from Lemma 6. Since $\mathrm{ch}_1(B_i) = 0$, we have that $B_i \cong A'_i \times B'_i$, where A'_i is an atomic Boolean algebra, and B'_i is an atomless Boolean algebra or $\mathbf{0}$.

Note that if $\exists^\infty i \ B'_i \cong \mathbf{0}$, then $\exists^\infty i \ B'_i \cong B_\eta$, since otherwise $\mathrm{ch}_1(B) = 0$. Let $B^1_i = B_i \times B_\eta$. It is clear that $B \cong \sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} B^1_i$. Hence we assume from now on that $B_i \cong A'_i \times B'_i$, where $B'_i \cong B_\eta$ for all i. Consider the following cases.

Case 1. $\exists^\infty i \ A'_i$ is infinite.

Case 1.1. There exists infinitely many i such that A'_i has a direct summand isomorphic to B_ω. Let $B^1_i = B_i \times A^*$, where A^* is a decidable presentation of B_ω. It is not hard to see that $B \cong \sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} B^1_i$. Thus, in this case we may assume that A'_i is infinite for all i.

Case 1.2. There are only finitely many i such that A'_i is infinite and has a direct summand isomorphic to B_ω. In this case $\exists^\infty i \ A'_i \cong B_{\omega\times\eta}$. Similarly to

Case 1.1, let $B_i^1 = B_i \times A^*$, where A^* is a decidable presentation of $B_{\omega\times\eta}$. It is not hard to see that $B \cong \sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} B_i^1$. Thus, in this case we may assume that A_i' is infinite for all i.

Case 2. $\exists^{<\infty} i\ A_i'$ is infinite. Collect all infinite A_i''s into a separate direct summand. By Lemma 5, the remaining part will be isomorphic to $B_{\omega+\eta}$, i.e., B is isomorphic to the direct sum of $B_{\omega+\eta}$ a computable infinite atomic Boolean algebra. Now the proof of Theorem 3 follows from Lemmas 7 and 8 below.

Lemma 7. *For every Π_2^0 set C, there exists a a computable Boolean algebra $B \cong B_{\omega+\eta}$ such that $Al(B) \equiv_T C$.*

Proof. If C is computable, then let B be a decidable presentation of $B_{\omega+\eta}$. Now, assume that C is not computable. Since C is a Π_2^0-set, there exists a computable predicate $R(x, s)$ such that

$$x \in C \iff \exists^{\infty} s\ R(x, s).$$

Let D be a computable atomless Boolean algebra and let $\{D_i\}_{i\in\omega}$ be a stringly computable sequence of finite subalgebras of D such that $D_0 = \{\mathbf{0}, \mathbf{1}\}$, $D_{i+1} = \mathrm{gr}(D_i \cup \{a_i\})$, where a_i is an atom of D_{i+1} and $D = \bigcup_{i\in\omega} D_i$.

Consider a computable sequence of Boolean algebras $\{B_i\}_{i\in\omega}$ such that $B_{2k} = D_0$ and $B_{2k+1} = D$ for all k. Construct a computable sequence $\{B_i'\}_{i\in\omega}$ step-by-step.

Step 0. For every k, let $B_{2k}^0 = B_{2k}$, $B_{2k+1}' = B_{2k+1}$.

Step $s + 1$. For all $k \leqslant s + 1$ such that $R(k, s + 1)$ do the following: if $B_{2k}^s = D_i$, then let $B_{2k}^{s+1} = D_{i+1}$. For all other k, let $B_{2k}^{s+1} = B_{2k}^s$. This concludes the step $s + 1$.

Let $B_{2k}' = \bigcup_{s\in\omega} B_{2k}^s$. Thus, the sequence $\{B_i'\}_{i\in\omega}$ is constructed. Let B be a natural computable presentation of $\sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} B_i'$. We have the following equivalence

$$k \in C \iff x^k \text{ is an atomless element of } \sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} B_i',$$

where

$$x^k(i) = \begin{cases} \mathbf{0}^{B_i'}, & \text{if } i \neq 2k, \\ \mathbf{1}^{B_i'}, & \text{if } i = 2k. \end{cases}$$

Hence, $C \leqslant_T Al(B)$. Furthermore, x is an atomless element of $\sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} B_i'$ iff there exists i_0 such that for all $i > i_0$, we have $x(i) = \mathbf{0}^{B_i'}$, and for all $i \leqslant i_0$,

$$i \text{ is even} \implies x(i) = \mathbf{0}^{B_i'} \text{ or } i/2 \in C.$$

Thus, $Al(B) \leqslant_T C$. Since C is not computable, $\mathbb{N} \setminus C$ is infinite. By Lemma 5, we have that $B \cong B_{\omega+\eta}$. □

Lemma 8. *Let $\{B_i\}_{i\in\omega}$ be a computable sequence of Boolean algebras such that $B_i \cong A'_i \times B'_i$, where A'_i is an infinite atomic Boolean algebra, $B'_i \cong B_\eta$, and the sets of atoms and atomless elements in B_i are uniformly computable in i. Then for every Π^0_2-set C, there exists a computable Boolean algebra $B \cong \sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} B_i$ such that $Al(B) \equiv_T C$.*

Proof. Since the sets of atoms in B_i's are uniformly computable in i, we can construct a computable sequence $\{a_i\}_{i\in\omega}$ such that a_i is an atom of B_i. Let D be the computable atomless Boolean algebra and let $\{D_i\}_{i\in\omega}$ be the strongly computable sequence of finite subalgebras as defined in the proof of Lemma 7. Consider the computable sequence $\{C_i\}_{i\in\omega}$ such that $C_{2k} = D_0$ and $C_{2k+1} = (\widehat{C(a_k)})_{B_k}$. It is clear that

$$\sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} C_i \cong \sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} B_i.$$

Let $R(x,s)$ be a computable predicate such that

$$x \in C \iff \exists^\infty s\ R(x,s).$$

Construct a computable sequence $\{C'_i\}_{i\in\omega}$ step-by-step.

Step 0. For every k, let $C^0_{2k} = C_{2k}$ and $C'_{2k+1} = C_{2k+1}$.

Step $s+1$. For every $k \leqslant s+1$ such that $R(k, s+1)$ do the following: if $C^s_{2k} = D_i$, then let $C^{s+1}_{2k} = D_{i+1}$. For all other k, let $C^{s+1}_{2k} = C^s_{2k}$. This concludes the step $s+1$.

Let $C'_{2k} = \bigcup_{s\in\omega} C^s_{2k}$. Since for every k, C'_{2k} is either a finite or the infinite atomless Boolean algebra, we have that $C'_{2k} \times C'_{2k+1} \cong B_k$. Hence,

$$\sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} C'_i \cong \sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} B_i.$$

Let B be a natural computable representation of $\sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} C'_i$. Since

$$k \in C \iff x^k \text{ is an atomless element of } \sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} C'_i,$$

where

$$x^k(i) = \begin{cases} \mathbf{0}^{C'_i}, & \text{if } i \neq 2k, \\ \mathbf{1}^{C'_i}, & \text{if } i = 2k, \end{cases}$$

we see that $C \leqslant_T Al(B)$. Note that $x \in \sum_{i\in\omega}{}_{\{\mathbf{0},\mathbf{1}\}} C'_i$ is atomless if and only if there exists i_0 such that the following conditions are satisfied:

1) for every $i > i_0$, $x(i) = \mathbf{0}^{C'_i}$,

2) for every odd $i \leqslant i_0$, $x(i)$ is an atomless element of B_k and $x(i) \leqslant C(a_k)$, where $k = (i-1)/2$,

3) for every even $i \leqslant i_0$, $x(i) = \mathbf{0}^{C'_i}$ or $i/2 \in C$.

Hence, $Al(B) \leqslant_T C$. □

Therefore, Theorem 3 is proved.

§3. Degree spectra of the sets of atoms. In this section we will study some properties of the degree spectra of the sets of atoms in computable Boolean algebras. First, we will need the following isomorphism theorem.

Theorem 9 (Isomorphism Theorem)**.** *Let A be a subalgebra of a Boolean algebra B such that:*

1) the set $Atom(A)$ of atoms of A is infinite;

2) if $a \in Atom(A)$, then $a \in Fr(B)$;

3) if $a \in Al(A)$, then $a \in S(B)$;

4) $B = \mathrm{gr}(A \cup Atom(B))$.

Then A and B are isomorphic.

Proof. Given $x, y \in B$, we write $x \sim y$ when $x \triangle y \in Fr(B)$. Since $B = \mathrm{gr}(A \cup Atom(B))$, we have that $\forall b \in B\ \exists a \in A\ \ a \sim b$. It is easy to see that $\forall a \in A\ (a \in S(A) \iff a \in S(B))$. Let

$$S = \{(a,b) \in A \times B : a \in Fr(A) \iff b \in Fr(B),\ a \in S(A) \iff b \in S(B),\ a \in S(A) \Longrightarrow At_A(a) = At_B(b),\ a \notin S(A) \Longrightarrow a \sim b\}.$$

A routine check shows that S is a condition of isomorphism for Boolean algebras A and B. Therefore, by Vaught's Criterion A and B are isomorphic. The theorem is proved. □

Theorem 10. *Let B be a computable Boolean algebra with infinitely many atoms such that $Fr(B), Al(B) \in \Delta^0_2$. Then there exists a computable Boolean algebra $A \cong B$ such that $Fr(A)$ is computably enumerable.*

Proof. Since B is computable, there exist a computably enumerable tree D and a partially computable function φ such that $\langle D, \varphi \rangle$ is a tree generating B. Also, there exists a strongly computable sequence $\{D_s\}_{s\in\omega}$ of finite subtrees of D such that $D = \bigcup_{s\in\omega} D_s$ and $D_{s+1} = D_s \cup \{L(a), R(a)\}$, where a is a leaf of D_s.

Call a vertex $x \in D$ *finite* if $\widehat{x} \cap D$ is finite, where $\widehat{x} = \{y : y \preccurlyeq x\}$. Call a vertex $x \in D$ *complete* if $\widehat{x} \subseteq D$. It is clear that

$$x \text{ is a finite vertex of } D \iff \varphi(x) \in Fr(B),$$
$$x \text{ is a complete vertex of } D \iff \varphi(x) \in Al(B).$$

Therefore, the sets of finite and complete vertices are Δ^0_2-sets, and hence there exist strongly computable sequences $\{F_s\}_{s\in\omega}$ and $\{G_s\}_{s\in\omega}$ of finite sets such that

$$x \text{ is a finite vertex of } D \Longrightarrow \forall^{\#} s\ x \in F_s,$$
$$x \text{ is not a finite vertex of } D \Longrightarrow \forall^{\#} s\ x \notin F_s,$$
$$x \text{ is a complete vertex of } D \Longrightarrow \forall^{\#} s\ x \in G_s,$$
$$x \text{ is not a complete vertex of } D \Longrightarrow \forall^{\#} s\ x \notin G_s.$$

Construct a new strongly computable sequence $\{F'_s\}_{s\in\omega}$ of finite sets such that

1) $F'_s \subseteq D_s$,
2) if x is a finite vertex of D, then $\forall^{\#} s\ x \in F'_s$,
3) if x is not a finite vertex of D, then $\forall^{\#} s\ x \notin F'_s$,
4) F'_s is a lower cone in D_s, i.e., for all $x, y \in D_s$ if $y \preccurlyeq x$ and $x \in F'_s$, then $y \in F'_s$,
5) $0 \notin F'_s$,
6) if $x \in D_s \setminus F'_s$, then there exists a leaf $y \preccurlyeq x$ of D_s such that $y \notin F'_s$,
7) if x is a complete vertex of D, then $\forall^{\#} s\ \widehat{x} \cap F'_s = \varnothing$.

Intuitively, these conditions mean that the sequence $\{F'_s\}_{s\in\omega}$ possesses the same properties as $\{F \cap D_s\}_{s\in\omega}$, where F is the set of all finite vertices of D.

Let $F'_0 = F_{s_0} \cap D_0$, where s_0 is the first step such that

a) $F_{s_0} \cap D_0$ is a lower cone in D_0,
b) $0 \notin F_{s_0} \cap D_0$,
c) if $x \in D_0 \setminus F_{s_0}$, then there exists a leaf $y \preccurlyeq x$ of D_0 such that $y \notin F_{s_0}$,
d) if $x \in G_{s_0}$, then $\widehat{x} \cap F_{s_0} \cap D_0 = \varnothing$.

Such s_0 always exists. Next, let $F'_1 = F_{s_1} \cap D_1$, where s_1 is the first step after s_0 at which the conditions a)–d) are satisfied after replacing F_{s_0}, G_{s_0} and D_0 with F_{s_1}, G_{s_1} and D_1, respectively. And so on.

As one can see, $\{F'_s\}_{s\in\omega}$ possesses the properties 1)–7). For convenience, we will write F_s instead of F'_s.

We will construct the required Boolean algebra A step-by-step. At the end of step s we will have a finite Boolean algebra A_s, a subtree $\widetilde{D}_s \subseteq D_s$, a map $f_s : \widetilde{D}_s \longrightarrow A_s$ such that $\langle \widetilde{D}_s, f_s \rangle$ is a tree generating Boolean algebra $\widetilde{A}_s = \mathrm{gr}(\{f_s(x) : x \in \widetilde{D}_s\})$, and finite sets Fr_s and Fr_s^-.

The domain of A_s will be an initial segment of $\mathbb{N}$. When we say in the construction "split an atom $a \in A_s$ into two atoms a_0 and a_1 in A_{s+1}", this means that we construct a Boolean algebra A_{s+1} such that the domain of A_{s+1} is an initial segment of $\mathbb{N}$, and $A_{s+1} = \mathrm{gr}(A_s \cup \{a_0\})$ with $a_0 \notin A_s$, $a_0 \leqslant a$, and $a_1 = a \backslash a_0$. Note that given A_s and an atom $a \in A_s$, this construction can be done effectively.

Let $f = \lim_s f_s$ and $\widetilde{A} = \mathrm{gr}(\{f(x) : x \in D\})$. For every $m > 0$, consider the following requirements:

R^0_m : if $m \in D$ is not a finite vertex, then $m \in \mathrm{dom}(f)$ and $f(m) \notin Fr(A)$,

R^1_m : if $m \in D$ is a finite vertex, then $m \in \mathrm{dom}(f)$ and $f(m) \in Fr(A)$.

Define the priority of the requirements as follows:

(1) if $n < m$ and $m \neq S(n)$, then $R^i_n > R^j_m$ for all $i, j \in \{0, 1\}$,

(2) if $n < m$ and $m = S(n)$, then $R^0_n > R^0_{S(n)} > R^1_n > R^1_{S(n)}$.

Description of the construction

Step 0. Let $A_0 = \{0, 1\}$ with 0 being the least and 1 being the the greatest elements of A_0, $\widetilde{D}_0 = \{0\}$, $f_0(0) = 1$, $Fr_0 = \varnothing$, $Fr^-_0 = \varnothing$.

Step $s + 1$. We say that

(i) the requirement R^0_m *attracts attention at step* $s+1$, if $m \in D_{s+1}$, $m \notin \widetilde{D}_s$, $H(m) \in \widetilde{D}_s$, $m \notin F_{s+1}$ or $m \in \widetilde{D}_s$, $m \notin F_{s+1}$, $f_s(m) \in Fr_s$ and there exists a leaf $k \preccurlyeq S(m)$ of $\widetilde{D}_s$ such that $f_s(k) \notin Fr_s$;

(ii) the requirement R^1_m *attracts attention at step* $s+1$, if $m \in D_{s+1}$, $m \notin \widetilde{D}_s$, $H(m) \in \widetilde{D}_s$, $m \in F_{s+1}$ or $m \in \widetilde{D}_s$, $m \in F_{s+1}$, $f_s(m) \notin Fr_s$.

Let R be the requirement of the highest priority that attracts attention at step $s + 1$. We say that R *acts* at step $s + 1$. Depending on the type of R, we proceed as follows:

(1) Suppose that $R = R^0_m$ and $m \in D_{s+1}$, $m \notin \widetilde{D}_s$, $H(m) \in \widetilde{D}_s$, $m \notin F_{s+1}$. Consider $f_s(H(m))$. If it is an atom of A_s, then split it into two atoms a_0 and a_1 in A_{s+1}. If $f_s(H(m)) = a \vee b$, where a is an atom of A_s and $b \in Fr^-_s$, then split a into two atoms a_0 and a_1 in A_{s+1}. Let $\widetilde{D}_{s+1} = \widetilde{D}_s \cup \{m, S(m)\}$, $f_{s+1} \upharpoonright \widetilde{D}_s = f_s$, $f_{s+1}(m) = a_0$, $f_{s+1}(S(m)) = f_s(H(m)) \backslash a_0$, $Fr_{s+1} = Fr_s$, $Fr^-_{s+1} = Fr^-_s$.

(2) Suppose that $R = R^0_m$ and $m \in \widetilde{D}_s$, $m \notin F_{s+1}$, $f_s(m) \in Fr_s$, and there exists a leaf $k \preccurlyeq S(m)$ of $\widetilde{D}_s$ such that $f_s(k) \notin Fr_s$. If $f_s(k)$ is an atom of A_s, then split it into two atoms a_0 and a_1 in A_{s+1}. If $f_s(k) = a \vee b$, where a is an atom of A_s and $b \in Fr^-_s$, then split a into two atoms a_0 and a_1 in A_{s+1}. Let $\widetilde{D}_{s+1} = \widetilde{D}_s \setminus \{k : k \prec m\}$,

$$f_{s+1}(n) = \begin{cases} f_s(n), & \text{if } H(m) \preccurlyeq n \text{ or } n \text{ is incomparable with } k \text{ and } m \\ f_s(n) \vee a_0, & \text{if } n = m \\ f_s(n) \backslash a_0, & \text{if } k \preccurlyeq n \preccurlyeq S(m), \end{cases}$$

$Fr^-_{s+1} = Fr^-_s \setminus \{b \in Fr^-_s : b \leqslant f_s(m)\} \cup \{f_s(m)\}$, $Fr_{s+1} = Fr_s$.

(3) Suppose that $R = R^1_m$ and $m \in D_{s+1}$, $m \notin \widetilde{D}_s$, $H(m) \in \widetilde{D}_s$, $m \in F_{s+1}$. Consider $f_s(H(m))$. If it is an atom of A_s, then split it into two atoms a_0 and a_1 in A_{s+1}. If $f_s(H(m)) = a \vee b$, where a is an atom of A_s and $b \in Fr^-_s$, then split a into two atoms a_0 and a_1 in A_{s+1}. Let $\widetilde{D}_{s+1} = \widetilde{D}_s \cup \{m, S(m)\}$, $f_{s+1} \upharpoonright \widetilde{D}_s = f_s$, $f_{s+1}(m) = a_0$, $f_{s+1}(S(m)) = f_s(H(m)) \backslash a_0$, $Fr_{s+1} = \{x \in A_{s+1} : \exists y \in Fr_s \;\; x \leqslant y\} \cup \{f_{s+1}(m)\}$, $Fr^-_{s+1} = Fr^-_s$.

(4) Suppose that $R = R^1_m$ and $m \in \widetilde{D}_s$, $m \in F_{s+1}$, $f_s(m) \notin Fr_s$. Let $Fr_{s+1} = Fr_s \cup \{x \in A_s : x \leqslant f_s(m)\}$, $A_{s+1} = A_s$, $\widetilde{D}_{s+1} = \widetilde{D}_s$, $f_{s+1} = f_s$, $Fr^-_{s+1} = Fr^-_s$.

This concludes the step $s+1$. Now the proof of Theorem 10 follows from the series of lemmas below.

Lemma 11. *For each s the following conditions hold:*

1) If k is a leaf of $\widetilde{D}_s$, then $f_s(k) = a \vee b$, where a is an atom of A_s, $a \notin Fr^-_s$ and ($b \in Fr^-_s$ or $b = 0$),

2) Fr_s is a lower cone in A_s,

3) $\forall n \in \widetilde{D}_s \setminus \{0\}$ $(f_s(n) \in Fr_s$ & $f_s(S(n)) \in Fr_s) \Longrightarrow f_s(H(n)) \in Fr_s$,

4) $f_s(0) = 1 \notin Fr_s$,

5) If k is a leaf of $\widetilde{D}_s$ and all the atoms of A_s below $f_s(k)$ are in Fr_s, then $f_s(k) \in Fr_s$,

6) $Fr^-_s \subseteq Fr_s$.

Proof. The proof is by induction on s. Suppose all these conditions hold at step s, and consider step $s+1$. Let R be the requirement that acts at this step. Consider the following cases:

(1) $R = R^0_m$ and $m \in D_{s+1}$, $m \notin \widetilde{D}_s$, $H(m) \in \widetilde{D}_s$, $m \notin F_{s+1}$. Since $m \notin F_{s+1}$, we have $H(m) \notin F_{s+1}$. By assumption $H(m) \in \widetilde{D}_s$. We want to show that $f_s(H(m)) \notin Fr_s$. Assume that $f_s(H(m)) \in Fr_s$. If there existed a leaf $k \preccurlyeq S(H(m))$ of $\widetilde{D}_s$ such that $f_s(k) \notin Fr_s$, then the requirement $R^0_{H(m)}$ would attract attention, which is impossible. Thus, for all leaves $k \in \widetilde{D}_s$ such that $k \preccurlyeq S(H(m))$ we have $f_s(k) \in Fr_s$. By the inductive hypothesis, we have that $f_s(H(H(m))) \in Fr_s$ but $H(H(m)) \notin F_{s+1}$. Repeating this argument a few more times, we have that $f_s(0) \in Fr_s$, which is a contradiction. Therefore, $f_s(H(m)) \notin Fr_s$. Now it is clear that all conditions hold.

(2) $R = R^0_m$ and $m \in \widetilde{D}_s$, $m \notin F_{s+1}$, $f_s(m) \in Fr_s$ and there exists a leaf $k \preccurlyeq S(m)$ of $\widetilde{D}_s$ such that $f_s(k) \notin Fr_s$. Let us check the condition 3). Let $k \preccurlyeq S(m)$ be a leaf of $\widetilde{D}_s$ such that $f_s(k) \notin Fr_s$. Then $f_s(k) = a_0 \vee a_1 \vee b$, where a_0, a_1 are atoms of A_{s+1}, $a_0 \vee a_1$ is an atom of A_s, and $b \in Fr^-_s$ or $b = 0$. Take $n \in \widetilde{D}_{s+1} \setminus \{0\}$ such that $f_{s+1}(n) \in Fr_{s+1}$ & $f_{s+1}(S(n)) \in Fr_{s+1}$. Then clearly $f_s(n) \in Fr_s$ and $f_s(S(n)) \in Fr_s$. By the inductive

hypothesis, we have that $f_s(H(n)) \in Fr_s$. Suppose that $f_{s+1}(H(n)) \notin Fr_{s+1}$. It is possible only in the case when $f_{s+1}(H(n)) = f_s(H(n)) \vee a_0$ or $f_{s+1}(H(n)) = f_s(H(n))\backslash a_0$. In the first case we have $H(n) = m$, which is impossible since m is a leaf of $\widetilde{D}_{s+1}$. In the second case, we have $k \preccurlyeq H(n)$. Then $f_s(k) \leqslant f_s(H(n)) \in Fr_s$ and $f_s(k) \in Fr_s$. This is a contradiction. Therefore, $f_{s+1}(H(n)) \in Fr_{s+1}$. It is now easy to check all the remaining conditions.

(3) $R = R^1_m$ and $m \in D_{s+1}$, $m \notin \widetilde{D}_s$, $H(m) \in \widetilde{D}_s$, $m \in F_{s+1}$. Let us check the condition 3), that is

$$\forall n \in \widetilde{D}_{s+1} \setminus \{0\} \; (f_{s+1}(n) \in Fr_{s+1} \;\&\; f_{s+1}(S(n)) \in Fr_{s+1} \\ \Longrightarrow f_{s+1}(H(n)) \in Fr_{s+1}).$$

Note that for every $n \in \widetilde{D}_s$, $f_s(n) \in Fr_s \Longleftrightarrow f_{s+1}(n) \in Fr_{s+1}$. Hence for every $n \in \widetilde{D}_s \setminus \{0\}$ the condition 3) holds. Let $n = m$. By the construction $f_{s+1}(m) \in Fr_{s+1}$. If $f_{s+1}(S(m)) = f_s(H(m))\backslash a_0 \in Fr_{s+1}$, then there exists $y \in Fr_s$ such that $f_s(H(m))\backslash a_0 \leqslant y$. Then $f_s(H(m)) \leqslant y$, and hence $f_s(H(m)) \in Fr_s$. Therefore, $f_{s+1}(H(m)) \in Fr_{s+1}$.

Let us check the condition 5). Consider the case when $k = S(m)$ since the other cases are trivial. Let $f_{s+1}(H(m)) = a_0 \vee a_1 \vee b$, $f_{s+1}(m) = a_0$, $f_{s+1}(S(m)) = a_1 \vee b$, where a_0, a_1 are atoms of A_{s+1}, $a_0 \vee a_1$ is an atom of A_s, and $b \in Fr^-_s$ or $b = 0$. Suppose that all the atoms of A_{s+1} below $f_{s+1}(S(m))$ are in Fr_{s+1}. Then $a_1 \in Fr_{s+1}$. Thus, there exists $y \in Fr_s$ such that $a_1 \leqslant y$. Hence $a = a_0 \vee a_1 \leqslant y$, and so $a \in Fr_s$. Moreover, all the atoms of A_{s+1} below b are atoms in A_s and belong to Fr_s. By the inductive hypothesis, $f_{s+1}(H(m)) \in Fr_s$ and therefore $f_{s+1}(S(m)) \in Fr_{s+1}$. It is now easy to check all the remaining conditions.

(4) $R = R^1_m$ and $m \in \widetilde{D}_s$, $m \in F_{s+1}$, $f_s(m) \notin Fr_s$. Consider only the condition 3) since the other conditions are trivial. We have to show that

$$\forall n \in \widetilde{D}_{s+1} \setminus \{0\} \; (f_{s+1}(n) \in Fr_{s+1} \;\&\; f_{s+1}(S(n)) \in Fr_{s+1} \\ \Longrightarrow f_{s+1}(H(n)) \in Fr_{s+1}).$$

Consider the case when $n = m$ since the other cases are clear. By the construction $f_{s+1}(m) \in Fr_{s+1}$. We want to prove that $f_{s+1}(S(m)) \notin Fr_{s+1}$. Suppose $f_{s+1}(S(m)) \in Fr_{s+1}$; then $f_{s+1}(S(m)) \in Fr_s$. Let us show that $S(m) \in F_{s+1}$. Indeed, assume $S(m) \notin F_{s+1}$; then there exists a leaf $k \preccurlyeq m$ of $\widetilde{D}_s$ such that $f_s(k) \notin Fr_s$ since otherwise the inductive hypothesis would imply that $f_s(m) \in Fr_s$. This contradicts the assumption that $f_s(m) \notin Fr_s$. Thus, we have $S(m) \in \widetilde{D}_s$, $S(m) \notin F_{s+1}$, $f_s(S(m)) = f_{s+1}(S(m)) \in Fr_s$ and there exists a leaf $k \preccurlyeq m = S(S(m))$ of $\widetilde{D}_s$ such that $f_s(k) \notin Fr_s$. Hence, the requirement $R^0_{S(m)}$ attracts attention at step $s + 1$. It is impossible since the requirement R^1_m acts at this step. This shows that

$S(m) \in F_{s+1}$. By assumption $m \in F_{s+1}$. Hence $H(m) \in F_{s+1}$. We have that $f_s(H(m)) \in Fr_s$ since otherwise the requirement $R^1_{H(m)}$ would attract attention at step $s+1$, which is impossible. Since $f_s(m) \leqslant f_s(H(m))$, we have that $f_s(m) \in Fr_s$. This contradicts the assumption that $f_s(m) \notin Fr_s$. Therefore, we have proved that $f_{s+1}(S(m)) \notin Fr_{s+1}$.

□

Lemma 12. *Each requirement acts only finitely often.*

Proof. Consider $n > 0$ such that $n+1 = S(n)$ and the requirements R^0_n, R^0_{n+1}, R^1_n and R^1_{n+1}. Let s_0 be the step after which no requirement of higher priority acts. Then there exists $s_1 \geqslant s_0$ such that $n, S(n) \in \widetilde{D}_s$ for all $s \geqslant s_1$. Furthermore, there exists $s_2 \geqslant s_1$ such that

$$\begin{aligned} n \text{ is a finite vertex } D &\Longrightarrow \forall s \geqslant s_2\ n \in F_s, \\ n \text{ is not a finite vertex } D &\Longrightarrow \forall s \geqslant s_2\ n \notin F_s, \\ S(n) \text{ is a finite vertex } D &\Longrightarrow \forall s \geqslant s_2\ S(n) \in F_s, \\ S(n) \text{ is not a finite vertex } D &\Longrightarrow \forall s \geqslant s_2\ S(n) \notin F_s. \end{aligned}$$

Consider the step s_2+1. Suppose that n and $S(n)$ are not finite vertices of D; then $n \notin F_{s_2}$. If $f_{s_2}(n) \in Fr_{s_2}$, then there exists a leaf $k \preccurlyeq S(n)$ of $\widetilde{D}_{s_2}$ such that $f_{s_2}(k) \notin Fr_{s_2}$ since otherwise $f_{s_2}(H(n)) \in Fr_{s_2}$ and $H(n) \notin F_{s_2}$. Since the requirement $R^0_{H(n)}$ does not attract attention at step s_2+1, for each leaf $k \preccurlyeq S(H(n))$ of $\widetilde{D}_{s_2}$, we have that $f_{s_2}(k) \in Fr_{s_2}$. Thus, $f_{s_2}(H(H(n)) \in Fr_{s_2}$ but $H(H(n)) \notin F_{s_2}$. Repeating this argument, we will eventually have that $f_{s_2}(0) \in Fr_{s_2}$, which is a contradiction. Hence, the requirement R^0_n attracts attention at step s_2+1, but then it acts at that step.

Therefore, $f_{s_2+1}(n) \notin Fr_{s_2+1}$, and similarly $f_{s_2+2}(S(n)) \notin Fr_{s_2+2}$. Thus, after the step s_2+2 none of the requirements R^0_n, R^0_{n+1}, R^1_n and R^1_{n+1} will ever attract attention, and so none of them will act.

Suppose that $S(n)$ is a finite but n is not a finite vertex of D. As before, we have that $f_{s_2+1}(n) \notin Fr_{s_2+1}$. If $f_{s_2+1}(S(n)) \notin Fr_{s_2+1}$, then the requirement $R^1_{S(n)}$ will attract attention at step s_2+2 and hence it will act at that step. Therefore, $f_{s_2+2}(S(n)) \in Fr_{s_2+2}$. Thus, after the step s_2+2 none of the requirements R^0_n, R^0_{n+1}, R^1_n and R^1_{n+1} will ever attract attention, and so none of them will act.

The cases when n is a finite but $S(n)$ is not a finite vertex of D and when n and $S(n)$ are finite vertices of D can be handled in a similar way. □

Lemma 13. *For every $n \in D$, there exists the limit $f(n) = \lim_s f_s(n)$.*

Proof. This is a direct corollary of the previous lemma. □

Let $\widetilde{A} = \mathrm{gr}(\{f(n) : n \in D\})$; then it is clear that $\langle D, f\rangle$ is a tree generating the Boolean algebra $\widetilde{A}$. Therefore, we have that $B \cong \widetilde{A}$. Let $Fr = \bigcup_{s\in\omega} Fr_s$. We will prove a few lemmas concerning the properties of $\widetilde{A}$ and Fr.

Lemma 14. *If a is an atom of $\widetilde{A}$, then $a \in Fr(A)$.*

Proof. Since a is an atom of $\widetilde{A}$, there exists a leaf n of D such that $f(n) = a$. Then there exists s_0 such that $f_s(n) = f(n)$ for all $s \geqslant s_0$. Consider the element $f_{s_0}(n) \in A_{s_0}$. We have that $f_{s_0}(n) = a \vee b$, where a is an atom of A_{s_0}, and $b \in Fr^-_{s_0}$ or $b = 0$. In either case $b \in Fr(A)$. Since n is a leaf of D and $f_s(n) = f_{s_0}(n)$ for all $s \geqslant s_0$, we have that a is an atom of A. Therefore, $a \in Fr(A)$. □

Lemma 15. *If n is a complete vertex of D, then $f(n) \in S(A)$.*

Proof. Consider s_0 such that $f_s(n) = f(n)$ for all $s \geqslant s_0$. Consider $s_1 \geqslant s_0$ such that $F_s \cap \widehat{n} = \varnothing$ for all $s \geqslant s_1$. There exists $s_2 \geqslant s_1$ such that $f_{s_2}(m) \notin Fr_{s_2}$ for all $m \in \widetilde{D}_{s_2} \cap \widehat{n}$. We now have the following equivalence: $x \leqslant f(n)$ and x is an atom of A if and only if $x \in A_{s_2}$ and there exists $b \in Fr^-_{s_2}$ such that $x \leqslant b \leqslant f(n)$. Clearly, $f(n)$ contains only finitely many atoms of A, i.e., $f(n) \in S(A)$. □

Lemma 16. *If x is an atomless element of $\widetilde{A}$, then $x \in S(A)$.*

Proof. This is a direct corollary of the previous lemma. □

Lemma 17. *If $x \in Fr$, then $x \in Fr(A)$.*

Proof. Let $x \in Fr$; consider s_0 such that $x \in Fr_{s_0}$. Suppose that $x = x_0 \vee \cdots \vee x_k$, where x_i is an atom of A_{s_0} for every $i = 1, \ldots, k$. Then all x_i's are in Fr_{s_0}. If there exists $b \in Fr^-_{s_0}$ such that $x_i \leqslant b$, then x_i is an atom of A. If there is no such b, then there exists a leaf n_i of $\widetilde{D}_{s_0}$ such that $f_{s_0}(n_i) = x_i \vee b$, where $b \in Fr^-_{s_0}$ or $b = 0$. If at a step $s \geqslant s_0$ some requirement R^0_m acts for $n_i \preccurlyeq m$, then x_i will be below an element of Fr^-_s; hence $x_i \in Fr(A)$. If no requirement of the form R^0_m for $n_i \preccurlyeq m$ acts after step s_0, then n_i is a finite vertex of D; hence $x_i \in Fr(A)$. Therefore, we have that $x \in Fr(A)$. □

Lemma 18. *If x is an atom of A, then $x \in Fr$.*

Proof. Consider s_0 such that x is an atom of A_{s_0}. If there exists $b \in Fr^-_{s_0}$ such that $x \leqslant b$, then $x \in Fr_{s_0}$. Otherwise, there exists a leaf n of $\widetilde{D}_{s_0}$ such that $f_{s_0}(n) = x \vee b$, where $b \in Fr^-_{s_0}$ or $b = 0$. If there exists $s_1 > s_0$ such that $f_{s_1}(n) \neq f_{s_0}(n)$, then it means that at some step greater than s_0 a requirement R^0_m for $n \preccurlyeq m$ has acted. In this case there exists $b' \in Fr^-_{s_1}$ such that $x \leqslant b'$. Thus, $x \in Fr_{s_1}$.

If $f_s(n) = f_{s_0}(n)$ for all $s \geqslant s_0$, then n is a leaf of D. Then there exists $s_1 \geqslant s_0$ such that $n \in F_s$ for all $s \geqslant s_1$. Furthermore, if $f_s(n) \notin Fr_s$, then the requirement R^1_n attracts attention at step s. Hence, there exists $s_2 \geqslant s_1$ such that $f_{s_2}(n) \in Fr_{s_2}$, and therefore $x \in Fr_{s_2}$. □

Lemma 19. *$Fr(A)$ is computably enumerable.*

Proof. Since

$$x \in Fr(A) \Longleftrightarrow \exists x_1 \dots \exists x_k \left(x = x_1 \vee \cdots \vee x_k \text{ and } \bigwedge_{i=1}^{k} x_i \in Fr \right)$$

and the set Fr is computably enumerable, then so is $Fr(A)$. □

Lemma 20. *A is generated by $\widetilde{A}$ and $Atom(A)$.*

Proof. Given $x, y \in A$, we will write $x \sim y$ if $x \triangle y \in Fr(A)$. Let $x \in A$, then there exists s_0 such that $x \in A_{s_0}$. Note that there exist leaves $n_1, \dots, n_k$ of $\widetilde{D}_{s_0}$ such that $x \sim f_{s_0}(n_1) \vee \cdots \vee f_{s_0}(n_k)$. Let $C_{s_0} = \{n_1, \dots, n_k\}$.

Suppose that at step $s \geqslant s_0$ we have constructed C_s, and consider the step $s+1$. If at this step a requirement of the form R^1_m or R^0_m of case (1) acts, then let $C_{s+1} = C_s$. If a requirement R^0_m of case (2) acts, then consider a leaf k of $\widetilde{D}_s$ from the definition of case (2). Let $C^{'}_s = C_s \setminus \widehat{m}$. If $k \preccurlyeq n \preccurlyeq S(m)$ for some $n \in C_s$, then let $C_{s+1} = C^{'}_s \cup \{m\}$; otherwise, let $C_{s+1} = C^{'}_s$.

It is clear that $x \sim \bigvee_{n \in C_s} f_s(n)$ for all $s \geqslant s_0$. Since we add to C_s only vertices of smaller levels, the limit $C = \lim_s C_s$ exists, and $x \sim \bigvee_{n \in C} f(n)$. Since $\bigvee_{n \in C} f(n) \in \widetilde{A}$, we have that $A = \mathrm{gr}(\widetilde{A} \cup Atom(A))$. □

By the Isomorphism Theorem (Theorem 9) we have that $A \cong \widetilde{A} \cong B$. By Lemma 19 $Fr(A)$ is computably enumerable. The theorem is proved. □

Theorem 21. *Let B be a computable Boolean algebra with infinitely many atoms such that $Fr(B)$ is computably enumerable. Then there exists a computable Boolean algebra $A \cong B$ such that $Atom(A)$ is computable.*

Proof. Consider a strongly computable sequence $\{B_i\}_{i \in \omega}$ of finite Boolean algebras such that $B_0 = \{\mathbf{0}, \mathbf{1}\}$, $B_{i+1} = \mathrm{gr}(B_i \cup \{a_i\})$, where a_i is an atom of B_{i+1}, and $B = \bigcup_{i \in \omega} B_i$, and also a strongly computable sequence $\{Fr_i\}_{i \in \omega}$ such that $Fr_0 = \varnothing$, $Fr_i \subseteq Fr_{i+1}$ and $Fr(B) = \bigcup_{i \in \omega} Fr_i$.

We construct the required Boolean algebra step-by-step. At step s we will construct a finite subalgebra A_s of Boolean algebra B_s and the set At_s consisting of atoms of A_s. Moreover, every atom of A_s not in At_s will be an atom of B_s.

Step 0. Let $A_0 = B_0$ and $At_0 = \varnothing$.

Step $s+1$. Let $B_{s+1} = \mathrm{gr}(B_s \cup \{a_s\})$, where a_s is an atom of B_{s+1}. Let c be an atom of A_s such that $a_s \leqslant c$. If $c \in At_s$, then let $A_{s+1} = A_s$; if $c \notin At_s$, then let $A_{s+1} = \mathrm{gr}(A_s \cup \{a_s\})$. Also let

$$At_{s+1} = \{x \in A_{s+1} : x \text{ is an atom of } A_{s+1} \text{ and } \exists y \in Fr_{s+1} \cap A_{s+1} \ (x \leqslant y)\}.$$

This concludes the step $s+1$.

Let $\widetilde{A} = \bigcup_{i \in \omega} A_i$ and $At = \bigcup_{i \in \omega} At_i$. Note that $Atom(\widetilde{A}) = At$. Indeed, if a is an atom of $\widetilde{A}$, then a is an atom of A_s for almost all s. If $a \notin At$, then a must be an atom of B since otherwise we would have split it. Therefore, $a \in Fr(B)$, and by the construction a will be enumerated into At_s at some step s.

Every atom of $\widetilde{A}$ is a union of finitely many atoms of B since $At \subseteq Fr(B)$. Suppose that b is an atom of B. Consider the first step s such that $b \in B_s$. Suppose that $b \leqslant a$, where a is an atom of A_s. If $a \in At_s$, then a is an atom of $\widetilde{A}$. If $a \notin At_s$, then $a = b$ and in this case a will also be an atom of $\widetilde{A}$. Therefore, each atom of B lies under some atom of $\widetilde{A}$, which implies that the set of atoms of $\widetilde{A}$ is infinite and $Al(\widetilde{A}) \subseteq Al(B) \subseteq S(B)$.

Let us show that $B = \mathrm{gr}(\widetilde{A} \cup Atom(B))$. Take $b \in B$ and consider the first step s such that $b \in B_s$. Then $b = b_1 \vee \ldots \vee b_k$, where $b_1, \ldots, b_k$ are atoms of B_s. Consider $a = a_1 \vee \ldots \vee a_k \in A_s$, where each a_i is an atom of A_s such that $b_i \leqslant a_i$. Note that $a \bigtriangleup b \in Fr(B)$. Therefore, $B = \mathrm{gr}(\widetilde{A} \cup Atom(B))$

Hence, by the Isomorphism Theorem (Theorem 9) we have that $\widetilde{A} \cong B$. Since $\widetilde{A}$ is a computably enumerable set, there exists a one-to-one computable function f such that $\rho f = \widetilde{A}$. Using f, define computable functions $\vee$, $\wedge$ and C on $\mathbb{N}$ such that $A = \langle \mathbb{N}, \vee, \wedge, C \rangle$ is a computable Boolean algebra isomorphic to $\widetilde{A}$. We have that $x \in Atom(A)$ iff $f(x) \in At$. Since At is computably enumerable, then so is $Atom(A)$. Therefore, it is computable. □

We now prove the main theorem about the Boolean algebras with 1-low set of atoms.

Theorem 22. *Let B be a computable Boolean algebra with 1-low set of atoms. Then there exists a computable Boolean algebra $A \cong B$ such that $Atom(A)$ is computable.*

PROOF. Note that $Fr(B)$ is a Σ^0_1-set with respect to $Atom(B)$, and $Al(B)$ is a Π^0_1-set with respect to $Atom(B)$. Since $Atom(B)$ is 1-low, we have that $Fr(B)$ and $Al(B)$ are Δ^0_2-sets. If B contains finitely many atoms, then the proof is trivial. If B contains infinitely many atoms, the proof follows from Theorems 10 and 21. □

After the author have proved Theorem 22, P.E. Alaev pointed out that the paper [7] by Knight and Stob contains the proof of the following statement: If a Boolean algebra B is a Δ^0_2-algebra with predicates distinguishing the set of atoms, the Fréchet ideal and the ideal of atomless elements, then there exists a Boolean algebra $A \cong B$ computable together with the predicate distinguishing the set of atoms. This fact can be used to obtain an alternative proof of Theorem 22.

§4. Open problems. The results from this paper raised the following open questions: Is the degree spectrum of the ideal of atomless elements in a computable Boolean algebra of characteristic $(1, 1, 0)$ or $(1, 0, 1)$ closed upwards? Does there exist a computable Boolean algebra of characteristic $(1, 1, 0)$ or $(1, 0, 1)$ whose ideal of atomless elements is intrinsically non-computable?

In view of Theorem 22 the following question arose: Suppose that the degree spectrum of the set of atoms in a computable Boolean algebra contains an n-low degree for some n. Does it then contain the computable degree?

Answering these questions could help to better understand the structure of the degree spectra of the sets of atoms and of atomless elements in computable Boolean algebras.

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